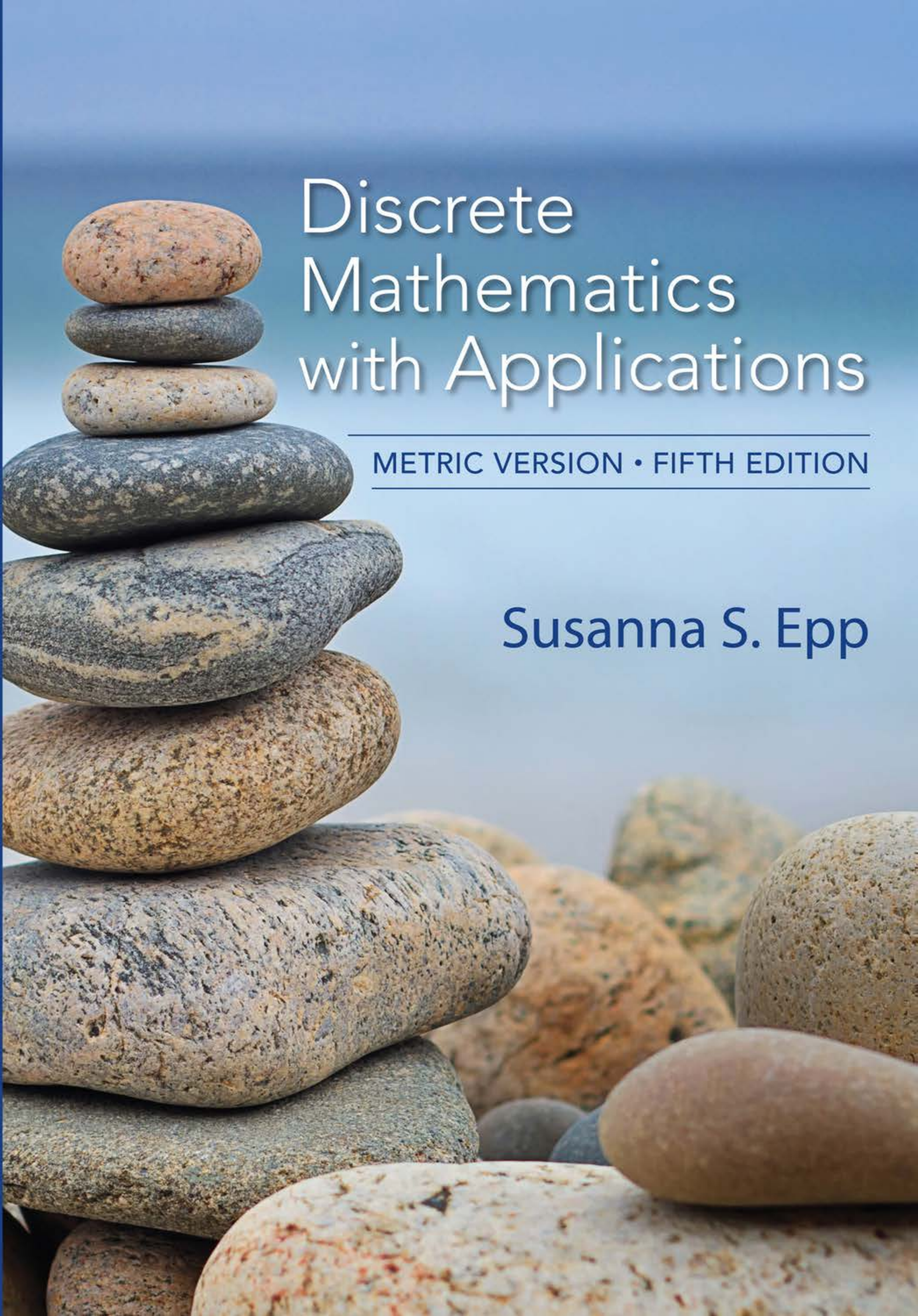
METRIC VERSION



Discrete Mathematics with Applications

METRIC VERSION • FIFTH EDITION

Susanna S. Epp

DISCRETE MATHEMATICS
WITH APPLICATIONS

FIFTH EDITION, METRIC VERSION

SUSANNA S. EPP
DePaul University

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Cover Photo: *The stones are discrete objects placed one on top of another like a chain of careful reasoning. A person who decides to build such a tower aspires to the heights and enjoys playing with a challenging problem. Choosing the stones takes both a scientific and an aesthetic sense. Getting them to balance requires patient effort and careful thought. And the tower that results is beautiful. A perfect metaphor for discrete mathematics!*

***Discrete Mathematics with Applications*, Fifth Edition, Metric Version**

Susanna S. Epp

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To my husband, Helmut, and my children,
Amanda, Catherine, and Caroline

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PREFACE

My purpose in writing this book was to provide a clear, accessible treatment of discrete mathematics for students majoring or minoring in computer science, mathematics, mathematics education, and engineering. The goal of the book is to lay the mathematical foundation for computer science courses such as data structures, algorithms, relational database theory, automata theory and formal languages, compiler design, and cryptography, and for mathematics courses such as linear and abstract algebra, combinatorics, probability, logic and set theory, and number theory. By combining discussion of theory and practice, I have tried to show that mathematics has engaging and important applications as well as being interesting and beautiful in its own right.

A good background in algebra is the only prerequisite; the course may be taken by students either before or after a course in calculus. Previous editions of the book have been used successfully by students at hundreds of institutions in North and South America, Europe, the Middle East, Asia, and Australia.

Recent curricular recommendations from the Institute for Electrical and Electronic Engineers Computer Society (IEEE-CS) and the Association for Computing Machinery (ACM) include discrete mathematics as the largest portion of "core knowledge" for computer science students and state that students should take at least a one-semester course in the subject as part of their first-year studies, with a two-semester course preferred when possible. This book includes the topics recommended by those organizations and can be used effectively for either a one-semester or a two-semester course.

At one time, most of the topics in discrete mathematics were taught only to upper-level undergraduates. Discovering how to present these topics in ways that can be understood by first- and second-year students was the major and most interesting challenge of writing this book. The presentation was developed over a long period of experimentation during which my students were in many ways my teachers. Their questions, comments, and written work showed me what concepts and techniques caused them difficulty, and their reaction to my exposition showed me what worked to build their understanding and to encourage their interest. Many of the changes in this edition have resulted from continuing interaction with students.

Themes of a Discrete Mathematics Course

Discrete mathematics describes processes that consist of a sequence of individual steps. This contrasts with calculus, which describes processes that change in a continuous fashion. Whereas the ideas of calculus were fundamental to the science and technology of the industrial revolution, the ideas of discrete mathematics underlie the science and technology of the computer age. The main themes of a first course in discrete mathematics are logic and proof, induction and recursion, discrete structures, combinatorics and discrete probability, algorithms and their analysis, and applications and modeling.

Logic and Proof Probably the most important goal of a first course in discrete mathematics is to help students develop the ability to think abstractly. This means learning to use logically valid forms of argument and avoid common logical errors, appreciating what it means to reason from definitions, knowing how to use both direct and indirect arguments to derive new results from those already known to be true, and being able to work with symbolic representations as if they were concrete objects.

Induction and Recursion An exciting development of recent years has been the increased appreciation for the power and beauty of "recursive thinking." To think recursively means to address a problem by assuming that similar problems of a smaller nature have already been solved and figuring out how to put those solutions together to solve the larger problem. Such thinking is widely used in the analysis of algorithms, where recurrence relations that result from recursive thinking often give rise to formulas that are verified by mathematical induction.

Discrete Structures Discrete mathematical structures are the abstract structures that describe, categorize, and reveal the underlying relationships among discrete mathematical objects. Those studied in this book are the sets of integers and rational numbers, general sets, Boolean algebras, functions, relations, graphs and trees, formal languages and regular expressions, and finite-state automata.

Combinatorics and Discrete Probability Combinatorics is the mathematics of counting and arranging objects, and probability is the study of laws concerning the measurement of random or chance events. Discrete probability focuses on situations involving discrete sets of objects, such as finding the likelihood of obtaining a certain number of heads when an unbiased coin is tossed a certain number of times. Skill in using combinatorics and probability is needed in almost every discipline where mathematics is applied, from economics to biology, to computer science, to chemistry and physics, to business management.

Algorithms and Their Analysis The word *algorithm* was largely unknown in the middle of the twentieth century, yet now it is one of the first words encountered in the study of computer science. To solve a problem on a computer, it is necessary to find an algorithm, or step-by-step sequence of instructions, for the computer to follow. Designing an algorithm requires an understanding of the mathematics underlying the problem to be solved. Determining whether or not an algorithm is correct requires a sophisticated use of mathematical induction. Calculating the amount of time or memory space the algorithm will need in order to compare it to other algorithms that produce the same output requires knowledge of combinatorics, recurrence relations, functions, and O-, Ω-, and Θ-notations.

Applications and Modeling Mathematical topics are best understood when they are seen in a variety of contexts and used to solve problems in a broad range of applied situations. One of the profound lessons of mathematics is that the same mathematical model can be used to solve problems in situations that appear superficially to be totally dissimilar. A goal of this book is to show students the extraordinary practical utility of some very abstract mathematical ideas.

Special Features of This Book

International Metric Version This metric version differs from the U.S. version of *Discrete Mathematics with Applications*, Fifth Edition as follows: The units of measurement used in the examples and exercises have been converted from the U.S. Customary System (USCS) of units (also referred to as English or Imperial units) to Metric units. The appendix containing solutions to selected exercises and the instructor's solutions manual have been converted to metric where appropriate.

Mathematical Reasoning The feature that most distinguishes this book from other discrete mathematics texts is that it teaches—explicitly but in a way that is accessible to first- and second-year college and university students—the unspoken logic and reasoning that underlie mathematical thought. For many years I taught an intensively interactive transition-to-abstract-mathematics course to mathematics and computer science majors. This experience showed me that while it is possible to teach the majority of students to understand and construct straightforward mathematical arguments, the obstacles to doing so cannot be passed over lightly. To be successful, a text for such a course must address students' difficulties with logic and language directly and at some length. It must also include enough concrete examples and exercises to enable students to develop the mental models needed to conceptualize more abstract problems. The treatment of logic and proof in this book blends common sense and rigor in a way that explains the essentials, yet avoids overloading students with formal detail.

Spiral Approach to Concept Development A number of concepts in this book appear in increasingly more sophisticated forms in successive chapters to help students develop the ability to deal effectively with increasing levels of abstraction. For example, by the time students encounter the relatively advanced mathematics of Fermat's little theorem in Section 8.4, they have been introduced to the logic of mathematical discourse in Chapters 1, 2, and 3, learned the basic methods of proof and the concepts of *mod* and *div* in Chapter 4, explored *mod* and *div* as functions in Chapter 7, and become familiar with equivalence relations in Sections 8.2 and 8.3. This approach builds in useful review and develops mathematical maturity in natural stages.

Support for the Student Students at colleges and universities inevitably have to learn a great deal on their own. Though it is often frustrating, learning to learn through self-study is a crucial step toward eventual success in a professional career. This book has a number of features to facilitate students' transition to independent learning.

Worked Examples
The book contains over 500 worked examples, which are written using a problem-solution format and are keyed in type and in difficulty to the exercises. Many solutions for the proof problems are developed in two stages: first a discussion of how one might come to think of the proof or disproof and then a summary of the solution, which is enclosed in a box. This format allows students to read the problem and skip immediately to the summary, if they wish, only going back to the discussion if they have trouble understanding the summary. The format also saves time for students who are rereading the text in preparation for an examination.

Marginal Notes and Test Yourself Questions
Notes about issues of particular importance and cautionary comments to help students avoid common mistakes are included in the margins throughout the book. Questions designed to focus attention on the main ideas of each section are located between the text and the exercises. For convenience, the questions use a fill-in-the-blank format, and the answers are found immediately after the exercises.

Exercises
The book contains almost 2600 exercises. The sets at the end of each section have been designed so that students with widely varying backgrounds and ability levels will find some exercises they can be sure to do successfully and also some exercises that will challenge them.

Solutions for Exercises
To provide adequate feedback for students between class sessions, Appendix B contains at least one, and often several, complete solutions for every type of exercise in the book. A blue exercise number indicates that there is a solution in Appendix B; the letter *H* is added for a solution that is less than complete. When two or more exercises use the same solution strategy, there is a full solution for the first and either another full solution or a partial solution for later ones. Exercises with several parts often have an answer and/or hint for one or more of the parts to help students determine whether they are on track so that they can make adjustments if needed.

Students are strongly urged not to consult solutions until they have tried their best to answer questions on their own. Once they have done so, however, comparing their answers with those given can lead to significantly improved understanding. There are also plenty of exercises without solutions to help students learn to grapple with mathematical problems in a realistic environment.

Reference Features
Many students have written me to say that the book helped them succeed in their advanced courses. One even wrote that he had used one edition so extensively that it had fallen apart, and he actually went out and bought a copy of the next edition, which he was continuing to use in a master's program. Figures and tables are included where doing so would help readers to a better understanding. In most, a second color is used to highlight meaning. My rationale for screening statements of definitions and theorems, for putting titles on exercises, and for giving the meanings of symbols and a list of reference formulas in the endpapers is to make it easier for students to use this book for review in a current course and as a reference in later ones.

Support for the Instructor I have received a great deal of valuable feedback from instructors who have used previous editions of this book. Many aspects of the book have been improved through their suggestions. In addition to the following items, there is additional instructor support on the book's website, described later in the preface.

Exercises
The large variety of exercises at all levels of difficulty allows instructors great freedom to tailor a course to the abilities of their students. Exercises with solutions in the back of the book have numbers in blue, and those whose solutions are given in a separate *Student Solutions Manual and Study Guide* have numbers that are a multiple of three. There are exercises of every type in the book that have no answer in either location so that instructors can assign whatever mixture they prefer of exercises with and without answers. The ample number of exercises of all kinds gives instructors a significant choice of problems to use for review assignments and exams. Instructors are invited to use the many exercises stated as questions rather than in "prove that" form to stimulate class discussion on the role of proof and counterexample in problem solving.

Flexible Sections
Most sections are divided into subsections so that an instructor can choose to cover certain subsections only and either omit the rest or leave them for students to study on their own. The division into subsections also makes it easier for instructors to break up sections if they wish to spend more than one day on them.

Presentation of Proof Methods
It is inevitable that most of the proofs and disproofs in this book will seem easy to instructors. Many students, however, find them difficult. In showing students how to discover and construct proofs and disproofs, I have tried to describe the kinds of approaches that mathematicians use when confronting challenging problems in their own research.

Complete Instructor Solutions
Complete instructor solutions to all exercises are available to anyone teaching a course from this book. They are available through the *Instructor's Companion Website*.

Highlights of the Fifth Edition

The changes made for this edition are based on suggestions from colleagues and other long-time users of previous editions, on continuing interactions with my students, and on developments within the evolving fields of computer science and mathematics.

Reorganization
- In response to instructor requests to move the introduction of certain topics earlier in the book, Section 1.2 now includes a definition and examples of strings. In addition, a new Section 1.4 contains definitions and examples of graphs and includes an introduction to graph coloring and the four-color theorem.
- The handshake theorem and its applications have been moved from Chapter 10 to Section 4.9. This gives students an early experience of using direct and indirect proof in a novel setting and was made possible because the elements of graph theory are now introduced in Chapter 1.

Improved Pedagogy
- The exposition has been reexamined throughout and carefully revised as needed.
- Exercises have been added for topics where students seemed to need additional practice, and they have been modified, as needed, to address student difficulties.
- Additional hints and full answers have been incorporated into Appendix B to give students more help for difficult topics.
- The introductory material in Chapter 4 was made more accessible by being divided into two sections. The first introduces basic concepts about proof and disproof in the context of elementary number theory, and the second adds examples and advice for writing proofs.

Logic and Applications
- Discussion was added about the role of bound variables and scope in mathematical writing and computer programming.
- The section on two's complements was significantly simplified.
- Language for expressing universal quantifiers was revised to provide a clearer basis for the idea of the generic particular in mathematical proof.
- The material on Boolean algebras was expanded.

Proof and Applications

- A greater variety of examples and exercises for number theory and set theory proofs is now included.
- The directions for writing proofs and the discussion of common mistakes have been revised and expanded in response to interaction with students.
- Discussion of historical background and recent mathematical results has been augmented.
- Material was added on using cryptographic hash functions to secure the transmission of digital data and on using cryptography to authenticate the sender of a transmitted message.

Induction and Recursion

- The sections on ordinary and strong mathematical induction were reorganized and expanded to increase the emphasis on applications.
- In the section on recursive definitions, the format used for proofs by structural induction was revised to parallel the format used for proofs by ordinary and strong mathematical induction. The set of examples and exercises illustrating recursive definitions and structural induction was significantly increased. The recursive definition for the set of strings over a finite set and for the length of a string were revised, and structural induction proofs for fundamental string properties are now included.

Graph Theory and the Analysis of Algorithm Efficiency

- Instructors who wish to give their students an early experience of graph theory can now do so by combining the introduction to graphs in Chapter 1 with the handshake theorem in Chapter 4.
- There is a new subsection on binary search trees in Chapter 10.
- The discussion of O-, Ω-, and Θ-notations was significantly simplified.
- Many exercises on algorithm efficiency were added or revised to make the concepts more accessible.

Student Resources

The *Student Companion Website* for this book includes:

- A general orientation for each chapter
- Review materials for each chapter
- Proof tips
- A link to the author's personal website, which contains errata information and links for interactive animations, tutorials, and other discrete mathematics resources on the Internet

Instructor's Resources

login.cengage.com

The *Instructor's Companion Website* for this book contains:

- Suggestions for how to approach the material of each chapter
- The *Complete Instructor's Solutions Manual*
- Ideas for projects and writing assignments
- Review materials to share with students
- Lecture Note PowerPoint slides
- Images from the book

- A test bank of questions for exams and quizzes
- Migration guide from 4th to 5th edition

Additional resources for the book are available at *http://condor.depaul.edu/sepp.*

WebAssign

www.webassign.com

WebAssign from Cengage *Discrete Mathematics with Applications,* Fifth Edition, Metric Version is an online homework system, which instructors can choose to pair with the book. For students, it offers tutorial help in solving exercises, including review of relevant material, short instructional videos, and instant feedback on how they are doing. For instructors, it offers the ability to create customized homework sets, most of which are graded automatically and produce results directly into an online grade roster. Real-time access to their students' performance makes it possible for instructors to adjust the presentation of material on an ongoing basis.

Student Solutions Manual and Study Guide

(ISBN: 978-0-357-03520-7)

In writing this book, I hoped that the exposition in the text, the worked examples, and the exercise solutions would provide all that a student would need to successfully master the material of the course. I continue to believe that any student who understands the solutions for all the exercises with complete solutions in Appendix B will have achieved an excellent command of the subject. Nonetheless, in response to requests for supplementary materials, I developed the *Student Solutions Manual and Study Guide*, available separately from the book, which contains complete solutions for all the exercises whose numbers are a multiple of 3. The guide also includes alternative explanations for some of the concepts and review questions for each chapter.

Organization

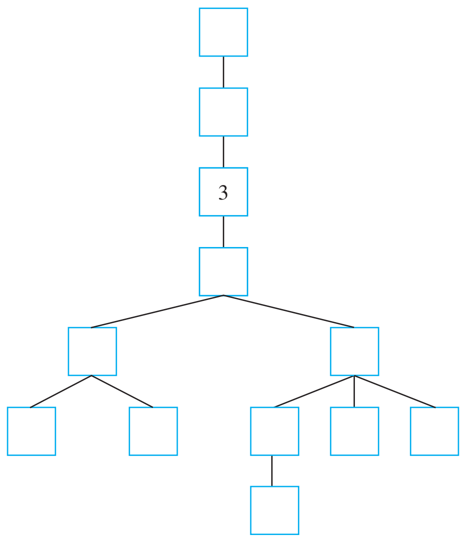
This book may be used effectively for a one- or two-semester course. Chapters contain core sections, sections covering optional mathematical material, and sections covering optional applications. Instructors have the flexibility to choose whatever mixture will best serve the needs of their students. The following table shows a division of the sections into categories.

Chapter	Core Sections	Sections Containing Optional Mathematical Material	Sections Containing Optional Computer Science Applications
1	1.1–1.3	1.4	1.4
2	2.1–2.3	2.5	2.4, 2.5
3	3.1–3.4	3.3	3.3
4	4.1–4.5, 4.7	4.6, 4.8, 4.9	4.10
5	5.1, 5.2, 5.6, 5.7	5.3, 5.4, 5.8	5.1, 5.5, 5.9
6	6.1	6.2–6.4	6.1, 6.4
7	7.1, 7.2	7.3, 7.4	7.1, 7.2, 7.4

8	8.1–8.3	8.4, 8.5	8.4, 8.5
9	9.1–9.4	9.5–9.9	9.3
10	10.1, 10.4	10.2, 10.3, 10.5	10.1, 10.4–10.6
11	11.1, 11.2	11.4	11.3, 11.5
12	12.1, 12.2	12.3	12.1–12.3

The following tree diagram shows, approximately, how the chapters of this book depend on each other. Chapters on different branches of the tree are sufficiently independent that instructors need to make at most minor adjustments if they skip chapters, or sections of chapters, but follow paths along branches of the tree.

In most cases, covering only the core sections of the chapters is adequate preparation for moving down the tree.



Acknowledgments

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My thanks go to the reviewers for their valuable suggestions for this edition of the book: Naser Al-Hasan, Newberry College; Linda Fosnaugh, Midwestern State Univer-

*Section 8.3 is needed for Section 12.3 but not for Sections 12.1 and 12.2.

sity; Robert Gessel, University of Akron; Juan Henriquez, University of New Orleans; Amy Hlavacek, Saginaw Valley State University; Kevin Lillis, Saint Ambrose University; Ramón Mata-Toledo, James Madison University; Bin Shao, University of San Francisco; Charles Qiao Zhang, Texas Christian University; and Cathleen Zucco-Teveloff, Rowan University. For their help with previous editions of the book, I am grateful to David Addis, Texas Christian University; Rachel Esselstein, California State University-Monterrey Bay; William Marion, Valparaiso University; Michael McClendon, University of Central Oklahoma; Steven Miller, Brown University; Itshak Borosh, Texas A & M University; Douglas M. Campbell, Brigham Young University; David G. Cantor, University of California at Los Angeles; C. Patrick Collier, University of Wisconsin-Oshkosh; Kevan H. Croteau, Francis Marion University; Irinel Drogan, University of Texas at Arlington; Pablo Echeverria, Camden County College; Henry A. Etlinger, Rochester Institute of Technology; Melvin J. Friske, Wisconsin Lutheran College; William Gasarch, University of Maryland; Ladnor Geissinger, University of North Carolina; Jerrold R. Griggs, University of South Carolina; Nancy Baxter Hastings, Dickinson College; Lillian Hupert, Loyola University Chicago; Joseph Kolibal, University of Southern Mississippi; Benny Lo, International Technological University; George Luger, University of New Mexico; Leonard T. Malinowski, Finger Lakes Community College; John F. Morrison, Towson State Unviersity; Paul Pederson, University of Denver; George Peck, Arizona State University; Roxy Peck, California Polytechnic State University, San Luis Obispo; Dix Pettey, University of Missouri; Anthony Ralston, State University of New York at Buffalo; Norman Richert, University of Houston-Clear Lake; John Roberts, University of Louisville; and George Schultz, St. Petersburg Junior College, Clearwater. Special thanks are due John Carroll, San Diego State University; Dr. Joseph S. Fulda; and Porter G. Webster, University of Southern Mississippi; Peter Williams, California State University at San Bernardino; and Jay Zimmerman, Towson University for their unusual thoroughness and their encouragement.

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I owe many thanks to the Cengage staff, especially my editors, Laura Gallus, Mona Zeftel, Lynh Pham, and Spencer Arritt, for their thoughtful advice and reassuringly calm direction of the production process, and my previous editors, Dan Seibert, Stacy Green, Robert Pirtle, Barbara Holland, and Heather Bennett, for their encouragement and enthusiasm.

The older I get the more I realize the profound debt I owe my own mathematics teachers for shaping the way I perceive the subject. My first thanks must go to my husband, Helmut Epp, who, on a high school date (!), introduced me to the power and beauty of the field axioms and the view that mathematics is a subject with ideas as well as formulas and techniques. In my formal education, I am most grateful to Daniel Zelinsky and Ky Fan at Northwestern University and Izaak Wirszup, I. N. Herstein, and Irving Kaplansky at the University of Chicago, all of whom, in their own ways, helped lead me to appreciate the elegance, rigor, and excitement of mathematics.

To my family, I owe thanks beyond measure. I am grateful to my mother, whose keen interest in the workings of the human intellect started me many years ago on the track that led ultimately to this book, and to my father, whose devotion to the written word has been a constant source of inspiration. I thank my children and grandchildren for their affection and cheerful acceptance of the demands this book has placed on my life. And, most of all, I am grateful to my husband, who for many years has encouraged me with his faith in the value of this project and supported me with his love and his wise advice.

Susanna Epp

CHAPTER 1 SPEAKING MATHEMATICALLY

Therefore O students study mathematics and do not build without foundations. —Leonardo da Vinci (1452–1519)

The aim of this book is to introduce you to a mathematical way of thinking that can serve you in a wide variety of situations. Often when you start work on a mathematical problem, you may have only a vague sense of how to proceed. You may begin by looking at examples, drawing pictures, playing around with notation, rereading the problem to focus on more of its details, and so forth. The closer you get to a solution, however, the more your thinking has to crystallize. And the more you need to understand, the more you need language that expresses mathematical ideas clearly, precisely, and unambiguously.

This chapter will introduce you to some of the special language that is a foundation for much mathematical thought, the language of variables, sets, relations, and functions. Think of the chapter like the exercises you would do before an important sporting event. Its goal is to warm up your mental muscles so that you can do your best.

1.1 Variables

A variable is sometimes thought of as a mathematical "John Doe" because you can use it as a placeholder when you want to talk about something but either (1) you imagine that it has one or more values but you don't know what they are, or (2) you want whatever you say about it to be equally true for all elements in a given set, and so you don't want to be restricted to considering only a particular, concrete value for it. To illustrate the first use, consider asking

> Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

In this sentence you can introduce a variable to replace the potentially ambiguous word "it":

> Is there a number x with the property that $2x + 3 = x^2$?

The advantage of using a variable is that it allows you to give a temporary name to what you are seeking so that you can perform concrete computations with it to help discover its possible values. To emphasize the role of the variable as a placeholder, you might write the following:

> Is there a number $\square$ with the property that $2 \cdot \square + 3 = \square^2$?

The emptiness of the box can help you imagine filling it in with a variety of different values, some of which might make the two sides equal and others of which might not.

In this sense, a variable in a computer program is similar to a mathematical variable because it creates a location in computer memory (either actual or virtual) into which values can be placed.

To illustrate the second use of variables, consider the statement

> No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

In this case introducing a variable to give a temporary name to the (arbitrary) number you might choose enables you to maintain the generality of the statement, and replacing all instances of the word "it" by the name of the variable ensures that possible ambiguity is avoided:

> No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Example 1.1.1 **Writing Sentences Using Variables**

Use variables to rewrite the following sentences more formally.

a. Are there numbers with the property that the sum of their squares equals the square of their sum?

b. Given any real number, its square is nonnegative.

Solution

Note In part (a) the answer is yes. For instance, $a = 1$ and $b = 0$ would work. Can you think of other numbers that would also work?

a. Are there numbers a and b with the property that $a^2 + b^2 = (a + b)^2$?
Or: Are there numbers a and b such that $a^2 + b^2 = (a + b)^2$?
Or: Do there exist any numbers a and b such that $a^2 + b^2 = (a + b)^2$?

b. Given any real number r, r^2 is nonnegative.
Or: For any real number r, $r^2 \geq 0$.
Or: For every real number r, $r^2 \geq 0$. ■

Some Important Kinds of Mathematical Statements

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

> A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero.*)
>
> A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6.*)
>
> Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even.*)

In later sections we will define each kind of statement carefully and discuss all of them in detail. The aim here is for you to realize that combinations of these statements can be expressed in a variety of different ways. One way uses ordinary, everyday language and another expresses the statement using one or more variables. The exercises are designed to help you start becoming comfortable in translating from one way to another.

Universal Conditional Statements

Universal statements contain some variation of the words "for every" and conditional statements contain versions of the words "if-then." A **universal conditional statement** is a statement that is both universal and conditional. Here is an example:

For every animal a, if a is a dog, then a is a mammal.

One of the most important facts about universal conditional statements is that they can be rewritten in ways that make them appear to be purely universal or purely conditional. For example, the previous statement can be rewritten in a way that makes its conditional nature explicit but its universal nature implicit:

If a is a dog, then a is a mammal.

Or: If an animal is a dog, then the animal is a mammal.

The statement can also be expressed so as to make its universal nature explicit and its conditional nature implicit:

For every dog a, a is a mammal.
Or: All dogs are mammals.

The crucial point is that the ability to translate among various ways of expressing universal conditional statements is enormously useful for doing mathematics and many parts of computer science.

Example 1.1.2 **Rewriting a Universal Conditional Statement**

Fill in the blanks to rewrite the following statement:

For every real number x, if x is nonzero then x^2 is positive.

a. If a real number is nonzero, then its square ______.
b. For every nonzero real number x, ______.
c. If x ______, then ______.
d. The square of any nonzero real number is ______.
e. All nonzero real numbers have ______.

Note If you introduce x in the first part of the sentence, be sure to include it in the second part of the sentence.

Solution

a. is positive
b. x^2 is positive
c. is a nonzero real number; x^2 is positive
d. positive
e. positive squares (*or*: squares that are positive) ■

Universal Existential Statements

A **universal existential statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example:

Every real number has an additive inverse.

Note For a number b to be an additive inverse for a number a means that $a + b = 0$.

In this statement the property "has an additive inverse" applies universally to all real numbers. "Has an additive inverse" asserts the existence of something—an additive inverse—for each real number. However, the nature of the additive inverse depends on the real number; different real numbers have different additive inverses. Knowing that an additive inverse is a real number, you can rewrite this statement in several ways, some less formal and some more formal:*

All real numbers have additive inverses.
Or: For every real number r, there is an additive inverse for r.
Or: For every real number r, there is a real number s such that s is an additive inverse for r.

Introducing names for the variables simplifies references in further discussion. For instance, after the third version of the statement you might go on to write: When r is positive, s is negative, when r is negative, s is positive, and when r is zero, s is also zero.

One of the most important reasons for using variables in mathematics is that it gives you the ability to refer to quantities unambiguously throughout a lengthy mathematical argument, while not restricting you to consider only specific values for them.

Example 1.1.3 Rewriting a Universal Existential Statement

Fill in the blanks to rewrite the following statement: Every pot has a lid.

a. All pots ______.

b. For every pot P, there is ______.

c. For every pot P, there is a lid L such that ______.

Solution

a. have lids

b. a lid for P

c. L is a lid for P ■

Existential Universal Statements

An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind. For example:

There is a positive integer that is less than or equal to every positive integer.

This statement is true because the number one is a positive integer, and it satisfies the property of being less than or equal to every positive integer. We can rewrite the statement in several ways, some less formal and some more formal:

Some positive integer is less than or equal to every positive integer.
Or: There is a positive integer m that is less than or equal to every positive integer.
Or: There is a positive integer m such that every positive integer is greater than or equal to m.
Or: There is a positive integer m with the property that for every positive integer n, $m \leq n$.

*A conditional could be used to help express this statement, but we postpone the additional complexity to a later chapter.

Example 1.1.4 **Rewriting an Existential Universal Statement**

Fill in the blanks to rewrite the following statement in three different ways:

> There is a person in my class who is at least as old as every person in my class.

a. Some ______ is at least as old as ______.

b. There is a person p in my class such that p is ______.

c. There is a person p in my class with the property that for every person q in my class, p is ______.

Solution

a. person in my class; every person in my class

b. at least as old as every person in my class

c. at least as old as q ■

Some of the most important mathematical concepts, such as the definition of limit of a sequence, can only be defined using phrases that are universal, existential, and conditional, and they require the use of all three phrases "for every," "there is," and "if-then." For example, if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, saying that

> the limit of a_n as n approaches infinity is L

means that

> **for every** positive real number ε, **there is** an integer N such that
> **for every** integer n, **if** $n > N$ **then** $-\varepsilon < a_n - L < \varepsilon$.

TEST YOURSELF

Answers to Test Yourself questions are located at the end of each section.

1. A universal statement asserts that a certain property is ______ for ______.
2. A conditional statement asserts that if one thing ______ then some other thing ______.
3. Given a property that may or may not be true, an existential statement asserts that ______ for which the property is true.

EXERCISE SET 1.1

Appendix B contains either full or partial solutions to all exercises with blue numbers. When the solution is not complete, the exercise number has an "*H*" next to it. A "*" next to an exercise number signals that the exercise is more challenging than usual. Be careful not to get into the habit of turning to the solutions too quickly. Make every effort to work exercises on your own before checking your answers. See the Preface for additional sources of assistance and further study.

In each of 1–6, fill in the blanks using a variable or variables to rewrite the given statement.

1. Is there a real number whose square is -1?
   a. Is there a real number x such that ______?
   b. Does there exist ______ such that $x^2 = -1$?
2. Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?
   a. Is there an integer n such that n has ______?
   b. Does there exist ______ such that if n is divided by 5 the remainder is 2 and if ______?

   Note: There are integers with this property. Can you think of one?
3. Given any two distinct real numbers, there is a real number in between them.

a. Given any two distinct real numbers a and b, there is a real number c such that c is ______.
b. For any two ______, ______ such that c is between a and b.

4. Given any real number, there is a real number that is greater.
a. Given any real number r, there is ______ s such that s is ______.
b. For any ______, ______ such that $s > r$.

5. The reciprocal of any positive real number is positive.
a. Given any positive real number r, the reciprocal of ______.
b. For any real number r, if r is ______, then ______.
c. If a real number r ______, then ______.

6. The cube root of any negative real number is negative.
a. Given any negative real number s, the cube root of ______.
b. For any real number s, if s is ______, then ______.
c. If a real number s ______, then ______.

7. Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.
a. There are real numbers u and v with the property that $u + v < u - v$.
b. There is a real number x such that $x^2 < x$.
c. For every positive integer n, $n^2 \geq n$.
d. For all real numbers a and b, $|a + b| \leq |a| + |b|$.

In each of 8–13, fill in the blanks to rewrite the given statement.

8. For every object J, if J is a square then J has four sides.
a. All squares ______.
b. Every square ______.
c. If an object is a square, then it ______.
d. If J ______, then J ______.
e. For every square J, ______.

9. For every equation E, if E is quadratic then E has at most two real solutions.
a. All quadratic equations ______.
b. Every quadratic equation ______.
c. If an equation is quadratic, then it ______.
d. If E ______, then E ______.
e. For every quadratic equation E, ______.

10. Every nonzero real number has a reciprocal.
a. All nonzero real numbers ______.
b. For every nonzero real number r, there is ______ for r.
c. For every nonzero real number r, there is a real number s such that ______.

11. Every positive number has a positive square root.
a. All positive numbers ______.
b. For every positive number e, there is ______ for e.
c. For every positive number e, there is a positive number r such that ______.

12. There is a real number whose product with every number leaves the number unchanged.
a. Some ______ has the property that its ______.
b. There is a real number r such that the product of r ______.
c. There is a real number r with the property that for every real number s, ______.

13. There is a real number whose product with every real number equals zero.
a. Some ______ has the property that its ______.
b. There is a real number a such that the product of a ______.
c. There is a real number a with the property that for every real number b, ______.

ANSWERS FOR TEST YOURSELF

1. true; all elements of a set 2. is true; also has to be true 3. there is at least one thing

1.2 The Language of Sets

. . . when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression. —George Polyá (1887–1985)

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as

Cantor did, simply as a collection of elements. For instance, if C is the set of all countries that are currently in the United Nations, then the United States is an element of C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

Set-Roster Notation

If S is a set, the notation $\boldsymbol{x \in S}$ means that x is an element of S. The notation $\boldsymbol{x \notin S}$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . . }** to refer to the set of all positive integers. (The symbol . . . is called an **ellipsis** and is read "and so forth.")

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

Example 1.2.1

Using the Set-Roster Notation

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Solution

a. A, B, and C have exactly the same three elements: 1, 2, and 3. Therefore, A, B, and C are simply different ways to represent the same set.

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

c. The set $\{1, \{1\}\}$ has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$. ■

Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the following table.

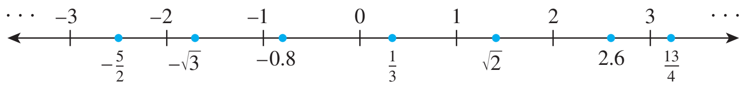
Symbol	Set
R	the set of all real numbers
Z	the set of all integers
Q	the set of all rational numbers, or quotients of integers

Note The **Z** is the first letter of the German word for integers, *Zahlen.* It stands for the *set* of all integers and should not be used as a shorthand for the word *integer.*

When the Symbols **R**, **Q**, and **Z** are handwritten, they appear as $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$.

Addition of a superscript + or − or the letters *nonneg* indicates that only the positive or negative or nonnegative elements of the set, respectively, are to be included. Thus $\mathbf{R}^+$ denotes the set of positive real numbers, and $\mathbf{Z}^{nonneg}$ refers to the set of nonnegative integers: 0, 1, 2, 3, 4, and so forth. Some authors refer to the set of nonnegative integers as the set of **natural numbers** and denote it as **N**. Other authors call only the positive integers natural numbers. To prevent confusion, we simply avoid using the phrase *natural numbers* in this book.

The set of real numbers is usually pictured as the set of all points on a line, as shown below. The number 0 corresponds to a middle point, called the *origin*. A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin. Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number. The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. *Note that 0 is neither positive nor negative.* Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called *continuous* because it is imagined to have no holes. The set of integers corresponds to a collection of points located at fixed intervals along the real number line. Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called *discrete*. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

Note We read the left-hand brace as "the set of all" and the vertical line as "such that." In all other mathematical contexts, however, we do not use a vertical line to denote the words "such that"; we abbreviate "such that" as "s. t." or "s. th." or "·∋·."

Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true.** We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all such that

Occasionally we will write $\{x \mid P(x)\}$ without being specific about where the element x comes from. It turns out that unrestricted use of this notation can lead to genuine contradictions in set theory. We will discuss one of these in Section 6.4 and will be careful to use this notation purely as a convenience in cases where the set S could be specified if necessary.

Example 1.2.2

Using the Set-Builder Notation

Given that $\mathbf{R}$ denotes the set of all real numbers, $\mathbf{Z}$ the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers, describe each of the following sets.

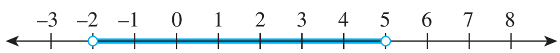
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$

Solution

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between -2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between -2 and 5. It is equal to the set $\{-1, 0, 1, 2, 3, 4\}$.

c. Since all the integers in $\mathbf{Z}^+$ are positive, $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}$. ■

Subsets

A basic relation between sets is that of subset.

Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$$A \subseteq B \quad \text{means that} \quad \text{for every element } x \text{, if } x \in A \text{ then } x \in B.$$

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

It follows from the definition of subset that for a set A not to be a subset of a set B means that there is at least one element of A that is not an element of B. Symbolically:

$$A \not\subseteq B \quad \text{means that} \quad \text{there is at least one element } x \text{ such that } x \in A \text{ and } x \notin B.$$

Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 1.2.3 Subsets

Let $A = \mathbf{Z}^+$, $B = \{n \in \mathbf{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

a. $B \subseteq A$

b. C is a proper subset of A

c. C and B have at least one element in common

d. $C \subseteq B$

e. $C \subseteq C$

Solution

a. False. Zero is not a positive integer. Thus zero is in B but zero is not in A, and so $B \not\subseteq A$.

b. True. Each element in C is a positive integer and, hence, is in A, but there are elements in A that are not in C. For instance, 1 is in A and not in C.

c. True. For example, 100 is in both C and B.

d. False. For example, 200 is in C but not in B.

e. True. Every element in C is in C. In general, the definition of subset implies that all sets are subsets of themselves. ■

Example 1.2.4 Distinction between ∈ and ⊆

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ b. $\{2\} \in \{1, 2, 3\}$ c. $2 \subseteq \{1, 2, 3\}$

d. $\{2\} \subseteq \{1, 2, 3\}$ e. $\{2\} \subseteq \{\{1\}, \{2\}\}$ f. $\{2\} \in \{\{1\}, \{2\}\}$

Solution Only (a), (d), and (f) are true.

For (b) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (b) is false.

For (c) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1, 2, 3\}$. This is not the case, so (c) is false.

For (e) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$ and $\{2\}$. But 2 is not equal to either $\{1\}$ or $\{2\}$, and so (e) is false. ■

Cartesian Products

Kazimierz Kuratowski (1896–1980)

With the introduction of Georg Cantor's set theory in the late nineteenth century, it began to seem possible to put mathematics on a firm logical foundation by developing all of its various branches from set theory and logic alone. A major stumbling block was how to use sets to define an ordered pair because the definition of a set is unaffected by the order in which its elements are listed. For example, $\{a, b\}$ and $\{b, a\}$ represent the same set, whereas in an ordered pair we want to be able to indicate which element comes first.

In 1914 crucial breakthroughs were made by Norbert Wiener (1894–1964), a young American who had recently received his Ph.D. from Harvard, and the German mathematician Felix Hausdorff (1868–1942). Both gave definitions showing that an ordered pair can be defined as a certain type of set, but both definitions were somewhat awkward. Finally, in 1921, the Polish mathematician Kazimierz Kuratowski (1896–1980) published the following definition, which has since become standard. It says that an ordered pair is a set of the form

$$\{\{a\}, \{a, b\}\}.$$

This set has elements, $\{a\}$ and $\{a, b\}$. If $a \neq b$, then the two sets are distinct and a is in both sets whereas b is not. This allows us to distinguish between a and b and say that a is the first element of the ordered pair and b is the second element of the pair. If $a = b$, then we can simply say that a is both the first and the second element of the pair. In this case the set that defines the ordered pair becomes $\{\{a\}, \{a, a\}\}$, which equals $\{\{a\}\}$.

However, it was only long after ordered pairs had been used extensively in mathematics that mathematicians realized that it was possible to define them entirely in terms of sets, and, in any case, the set notation would be cumbersome to use on a regular basis. The usual notation for ordered pairs refers to $\{\{a\}, \{a, b\}\}$ more simply as (a, b).

Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 1.2.5 **Ordered Pairs**

a. Is $(1, 2) = (2, 1)$?
b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?
c. What is the first element of $(1, 1)$?

Solution

a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2, 1) \text{ if, and only if, } 1 = 2 \text{ and } 2 = 1.$$

But $1 \neq 2$, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered pairs,

$$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right) \text{ if, and only if, } 3 = \sqrt{9} \text{ and } \frac{5}{10} = \frac{1}{2}.$$

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair $(1, 1)$, the first and the second elements are both 1. ■

The notation for an *ordered n-tuple* generalizes the notation for an ordered pair to a set with any finite number of elements. It also takes both order and multiplicity into account.

Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered *n*-tuple, $(x_1, x_2, \ldots, x_n)$**, consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots,$ and $x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

Example 1.2.6 **Ordered *n*-tuples**

a. Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?
b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Solution

a. No. By definition of equality of ordered 4-tuples,

$$(1, 2, 3, 4) = (1, 2, 4, 3) \;\Leftrightarrow\; 1 = 1, 2 = 2, 3 = 4, \text{ and } 4 = 3$$

But $3 \neq 4$, and so the ordered 4-tuples are not equal.

b. Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \frac{1}{2} = \frac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal. ■

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\boldsymbol{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example 1.2.7

Cartesian Products

Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$.

a. Find $A \times B$.

b. Find $B \times A$.

c. Find $A \times A$.

d. How many elements are in $A \times B$, $B \times A$, and $A \times A$?

e. Find $(A \times B) \times C$

f. Find $A \times B \times C$

g. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

Solution

a. $A \times B = \{(x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)\}$

b. $B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

c. $A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$

Note This is why it makes sense to call a Cartesian product a product!

d. $A \times B$ has 6 elements. Note that this is the number of elements in A times the number of elements in B. $B \times A$ has 6 elements, the number of elements in B times the number of elements in A. $A \times A$ has 4 elements, the number of elements in A times the number of elements in A.

e. The Cartesian product of A and B is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A \times B) \times C$.

$$\begin{aligned}(A \times B) \times C &= \{(u, v) \mid u \in A \times B \text{ and } v \in C\} \quad \text{by definition of Cartesian product} \\ &= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), \\ &\quad ((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), \\ &\quad ((y, 1), b), ((y, 2), b), ((y, 3), b)\}\end{aligned}$$

f. The Cartesian product $A \times B \times C$ is superficially similar to but is not quite the same mathematical object as $(A \times B) \times C$. $(A \times B) \times C$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A \times B \times C$ is a set of ordered triples. By definition of Cartesian product,

$$\begin{aligned} A \times B \times C &= \{(u, v, w) \mid u \in A, v \in B, \text{ and } w \in C\} \\ &= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a), (y, 3, a), (x, 1, b), \\ &\quad (x, 2, b), (x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b)\}. \end{aligned}$$

g. $\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs (x, y) where both x and y are real numbers. If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point. The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.

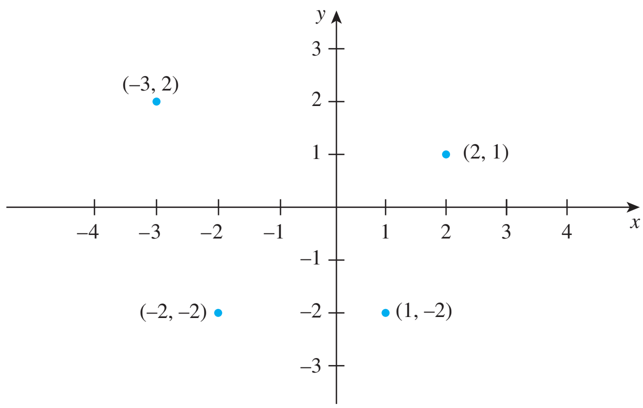


FIGURE 1.2.1 A Cartesian Plane ■

Another notation, which is important in both mathematics and computer science, denotes objects called *strings*.*

Definition

Let n be a positive integer. Given a finite set A, a **string of length n over A** is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters. It is often denoted λ and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example 1.2.8 **Strings**

Let $A = \{a, b\}$. List all the strings of length 3 over A with at least two characters that are the same.

Solution

$$aab, aba, baa, aaa, bba, bab, abb, bbb$$

In computer programming it is important to distinguish among different kinds of data structures and to respect the notations that are used for them. Similarly in mathematics, it is important to distinguish among, say, $\{a, b, c\}$, $\{\{a, b\}, c\}$, (a, b, c), $(a, (b, c))$, abc, and so forth, because these are all significantly different objects. ■

*A more formal definition of string, using recursion, is given in Section 5.9.

TEST YOURSELF

1. When the elements of a set are given using the set-roster notation, the order in which they are listed ______.
2. The symbol $\mathbf{R}$ denotes ______.
3. The symbol $\mathbf{Z}$ denotes ______.
4. The symbol $\mathbf{Q}$ denotes ______.
5. The notation $\{x \mid P(x)\}$ is read ______.
6. For a set A to be a subset of a set B means that ______.
7. Given sets A and B, the Cartesian product $A \times B$ is ______.
8. Given sets A, B, and C, the Cartesian product $A \times B \times C$ is ______.
9. A string of length n over a set S is an ordered n-tuple of elements of S, written without ______ or ______.

EXERCISE SET 1.2

1. Which of the following sets are equal?

$$A = \{a, b, c, d\} \qquad B = \{d, e, a, c\}$$
$$C = \{d, b, a, c\} \qquad D = \{a, a, d, e, c, e\}$$

2. Write in words how to read each of the following out loud.
   a. $\{x \in \mathbf{R}^+ \mid 0 < x < 1\}$
   b. $\{x \in \mathbf{R} \mid x \le 0 \text{ or } x \ge 1\}$
   c. $\{n \in \mathbf{Z} \mid n \text{ is a factor of } 6\}$
   d. $\{n \in \mathbf{Z}^+ \mid n \text{ is a factor of } 6\}$

3. a. Is $4 = \{4\}$?
   b. How many elements are in the set $\{3, 4, 3, 5\}$?
   c. How many elements are in the set $\{1, \{1\}, \{1, \{1\}\}\}$?

4. a. Is $2 \in \{2\}$?
   b. How many elements are in the set $\{2, 2, 2, 2\}$?
   c. How many elements are in the set $\{0, \{0\}\}$?
   d. Is $\{0\} \in \{\{0\}, \{1\}\}$?
   e. Is $0 \in \{\{0\}, \{1\}\}$?

H 5. Which of the following sets are equal?

$$A = \{0, 1, 2\}$$
$$B = \{x \in \mathbf{R} \mid -1 \le x < 3\}$$
$$C = \{x \in \mathbf{R} \mid -1 < x < 3\}$$
$$D = \{x \in \mathbf{Z} \mid -1 < x < 3\}$$
$$E = \{x \in \mathbf{Z}^+ \mid -1 < x < 3\}$$

H 6. For each integer n, let $T_n = \{n, n^2\}$. How many elements are in each of T_2, T_{-3}, T_1, and T_0? Justify your answers.

7. Use the set-roster notation to indicate the elements in each of the following sets.
   a. $S = \{n \in \mathbf{Z} \mid n = (-1)^k, \text{ for some integer } k\}$.
   b. $T = \{m \in \mathbf{Z} \mid m = 1 + (-1)^i, \text{ for some integer } i\}$.
   c. $U = \{r \in \mathbf{Z} \mid 2 \le r \le -2\}$
   d. $V = \{s \in \mathbf{Z} \mid s > 2 \text{ or } s < 3\}$
   e. $W = \{t \in \mathbf{Z} \mid 1 < t < -3\}$
   f. $X = \{u \in \mathbf{Z} \mid u \le 4 \text{ or } u \ge 1\}$

8. Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
   a. Is $B \subseteq A$?
   b. Is $C \subseteq A$?
   c. Is $C \subseteq C$?
   d. Is C a proper subset of A?

9. a. Is $3 \in \{1, 2, 3\}$?
   b. Is $1 \subseteq \{1\}$?
   c. Is $\{2\} \in \{1, 2\}$?
   d. Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
   e. Is $1 \in \{1\}$?
   f. Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
   g. Is $\{1\} \subseteq \{1, 2\}$?
   h. Is $1 \in \{\{1\}, 2\}$?
   i. Is $\{1\} \subseteq \{1, \{2\}\}$?
   j. Is $\{1\} \subseteq \{1\}$?

10. a. Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?
    b. Is $(5, -5) = (-5, 5)$?
    c. Is $(8 - 9, \sqrt[3]{-1}) = (-1, -1)$?
    d. Is $\left(\frac{-2}{-4}, (-2)^3\right) = \left(\frac{3}{6}, -8\right)$?

11. Let $A = \{w, x, y, z\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.
    a. $A \times B$
    b. $B \times A$
    c. $A \times A$
    d. $B \times B$

12. Let $S = \{2, 4, 6\}$ and $T = \{1, 3, 5\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.
a. $S \times T$
b. $T \times S$
c. $S \times S$
d. $T \times T$

13. Let $A = \{1, 2, 3\}$, $B = \{u\}$, and $C = \{m, n\}$. Find each of the following sets.
a. $A \times (B \times C)$
b. $(A \times B) \times C$
c. $A \times B \times C$

14. Let $R = \{a\}$, $S = \{x, y\}$, and $T = \{p, q, r\}$. Find each of the following sets.
a. $R \times (S \times T)$
b. $(R \times S) \times T$
c. $R \times S \times T$

15. Let $S = \{0, 1\}$. List all the strings of length 4 over S that contain three or more 0's.

16. Let $T = \{x, y\}$. List all the strings of length 5 over T that have exactly one y.

ANSWERS FOR TEST YOURSELF

1. does not matter **2.** the set of all real numbers **3.** the set of all integers **4.** the set of all rational numbers **5.** the set of all x such that $P(x)$ **6.** every element in A is an element in B **7.** the set of all ordered pairs (a, b) where a is in A and b is in B **8.** the set of ordered triples of the form (a, b, c) where $a \in A$, $b \in B$, and $c \in C$ **9.** parentheses; commas

1.3 The Language of Relations and Functions

Mathematics is a language. —Josiah Willard Gibbs (1839–1903)

There are many kinds of relationships in the world. For instance, we say that two people are related by blood if they share a common ancestor and that they are related by marriage if one shares a common ancestor with the spouse of the other. We also speak of the relationship between student and teacher, between people who work for the same employer, and between people who share a common ethnic background.

Similarly, the objects of mathematics may be related in various ways. A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common. A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$. Two identifiers in a computer program may be said to be related if they have the same first eight characters, or if the same memory location is used to store their values when the program is executed. And the list could go on!

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$ and let us say that an element x in A is related to an element y in B if, and only if, x is less than y. Let us use the notation $x\ R\ y$ as a shorthand for the sentence "x is related to y." Then

$$
\begin{array}{lll}
0\ R\ 1 & \text{since} & 0 < 1, \\
0\ R\ 2 & \text{since} & 0 < 2, \\
0\ R\ 3 & \text{since} & 0 < 3, \\
1\ R\ 2 & \text{since} & 1 < 2, \\
1\ R\ 3 & \text{since} & 1 < 3, \quad \text{and} \\
2\ R\ 3 & \text{since} & 2 < 3.
\end{array}
$$

On the other hand, if the notation $x \not R y$ represents the sentence "x is not related to y," then

$$\begin{aligned} 1 \not R 1 &\quad \text{since} \quad 1 \nless 1, \\ 2 \not R 1 &\quad \text{since} \quad 2 \nless 1, \quad \text{and} \\ 2 \not R 2 &\quad \text{since} \quad 2 \nless 2. \end{aligned}$$

Recall that the Cartesian product of A and B, $A \times B$, consists of all ordered pairs whose first element is in A and whose second element is in B:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

In this case,

$$A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

The elements of some ordered pairs in $A \times B$ are related, whereas the elements of other ordered pairs are not. Consider the set of all ordered pairs in $A \times B$ whose elements are related

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

Observe that knowing which ordered pairs lie in this set is equivalent to knowing which elements are related to which. The relation itself can therefore be thought of as the totality of ordered pairs whose elements are related by the given condition. The formal mathematical definition of relation, based on this idea, was introduced by the American mathematician and logician C. S. Peirce in the nineteenth century.

Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\, R\, y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.

The notation for a relation R may be written symbolically as follows:

$$\boldsymbol{x\, R\, y} \quad \text{means that} \quad \boldsymbol{(x, y) \in R}.$$

The notation $x \not R y$ means that x is not related to y by R:

$$\boldsymbol{x \not R y} \quad \text{means that} \quad \boldsymbol{(x, y) \notin R}.$$

Example 1.3.1

A Relation as a Subset

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$

a. State explicitly which ordered pairs are in $A \times B$ and which are in R.

b. Is $1\, R\, 3$? Is $2\, R\, 3$? Is $2\, R\, 2$?

c. What are the domain and co-domain of R?

Solution

a. $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$. To determine explicitly the composition of R, examine each ordered pair in $A \times B$ to see whether its elements satisfy the defining condition for R.

$$(1, 1) \in R \text{ because } \tfrac{1-1}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$
$$(1, 2) \notin R \text{ because } \tfrac{1-2}{2} = \tfrac{-1}{2}, \text{ which is not an integer.}$$
$$(1, 3) \in R \text{ because } \tfrac{1-3}{2} = \tfrac{-2}{2} = -1, \text{ which is an integer.}$$
$$(2, 1) \notin R \text{ because } \tfrac{2-1}{2} = \tfrac{1}{2}, \text{ which is not an integer.}$$
$$(2, 2) \in R \text{ because } \tfrac{2-2}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$
$$(2, 3) \notin R \text{ because } \tfrac{2-3}{2} = \tfrac{-1}{2}, \text{ which is not an integer.}$$

Thus

$$R = \{(1, 1), (1, 3), (2, 2)\}$$

b. Yes, $1\,R\,3$ because $(1, 3) \in R$.
No, $2 \not R\, 3$ because $(2, 3) \notin R$.
Yes, $2\,R\,2$ because $(2, 2) \in R$.

c. The domain of R is $\{1, 2\}$ and the co-domain is $\{1, 2, 3\}$. ■

Example 1.3.2 **The Circle Relation**

Define a relation C from $\mathbf{R}$ to $\mathbf{R}$ as follows: For any $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in C \quad \text{means that} \quad x^2 + y^2 = 1.$$

a. Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$? Is $-2\;C\;0$? Is $0\;C\;(-1)$? Is $1\;C\;1$?

b. What are the domain and co-domain of C?

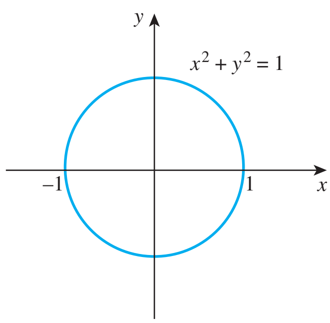
c. Draw a graph for C by plotting the points of C in the Cartesian plane.

Solution

a. Yes, $(1, 0) \in C$ because $1^2 + 0^2 = 1$.
No, $(0, 0) \notin C$ because $0^2 + 0^2 = 0 \neq 1$.
Yes, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$ because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$.
No, $-2 \not C\, 0$ because $(-2)^2 + 0^2 = 4 \neq 1$.
Yes, $0\;C\;(-1)$ because $0^2 + (-1)^2 = 1$.
No, $1 \not C\, 1$ because $1^2 + 1^2 = 2 \neq 1$.

b. The domain and co-domain of C are both $\mathbf{R}$, the set of all real numbers.

c.



■

Arrow Diagram of a Relation

Suppose R is a relation from a set A to a set B. The **arrow diagram for R** is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

Draw an arrow from x to y
if, and only if, $\quad x\,R\,y$
if, and only if, $\quad (x, y) \in R$.

Example 1.3.3 Arrow Diagrams of Relations

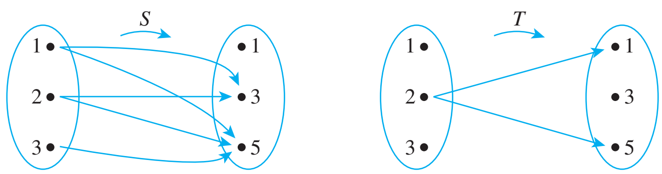
Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3\}$ and define relations S and T from A to B as follows: For every $(x, y) \in A \times B$,

$(x, y) \in S$ means that $x < y$ S is a "less than" relation.

$T = \{(2, 1), (2, 5)\}$.

Draw arrow diagrams for S and T.

Solution



These example relations illustrate that it is possible to have several arrows coming out of the same element of A pointing in different directions. Also, it is quite possible to have an element of A that does not have an arrow coming out of it. ■

Functions

In Section 1.2 we showed that ordered pairs can be defined in terms of sets and we defined Cartesian products in terms of ordered pairs. In this section we introduced relations as subsets of Cartesian products. Thus we can now define functions in a way that depends only on the concept of set. Although this definition is not obviously related to the way we usually work with functions in mathematics, it is satisfying from a theoretical point of view, and computer scientists like it because it is particularly well suited for operating with functions on a computer.

Definition

A **function F from a set A to a set B** is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Properties (1) and (2) can be stated less formally as follows: A relation F from A to B is a function if, and only if:

1. Every element of A is the first element of an ordered pair of F.
2. No two distinct ordered pairs in F have the same first element.

In most mathematical situations we think of a function as sending elements from one set, the domain, to elements of another set, the co-domain. Because of the definition of function, each element in the domain corresponds to one and only one element of the co-domain.

More precisely, if F is a function from a set A to a set B, then given any element x in A, property (1) from the function definition guarantees that there is at least one element of B that is related to x by F and property (2) guarantees that there is at most one such element. This makes it possible to give the element that corresponds to x a special name.

Function Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "***F* of *x***."

Example 1.3.4

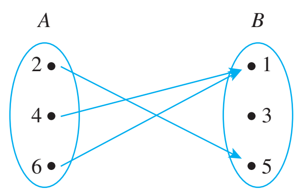
Functions and Relations on Finite Sets

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

a. $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$

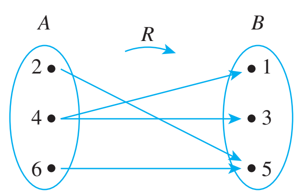
b. For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.

c. T is defined by the arrow diagram

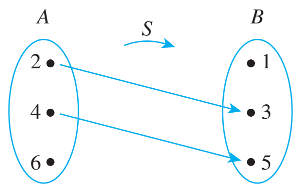


Solution

a. R is not a function because it does not satisfy property (2). The ordered pairs (4, 1) and (4, 3) have the same first element but different second elements. You can see this graphically if you draw the arrow diagram for R. There are two arrows coming out of 4: One points to 1 and the other points to 3.



b. S is not a function because it does not satisfy property (1). It is not true that every element of A is the first element of an ordered pair in S. For example, $6 \in A$ but there is no y in B such that $y = 6 + 1 = 7$. You can also see this graphically by drawing the arrow diagram for S.



Note In part (c), $T(4) = T(6)$. This illustrates the fact that although no element of the domain of a function can be related to more than one element of the co-domain, several elements in the domain can be related to the same element in the co-domain.

c. T is a function: Each element in $\{2, 4, 6\}$ is related to some element in $\{1, 3, 5\}$, and no element in $\{2, 4, 6\}$ is related to more than one element in $\{1, 3, 5\}$. When these properties are stated in terms of the arrow diagram, they become (1) there is an arrow coming out of each element of the domain, and (2) no element of the domain has more than one arrow coming out of it. So you can write $T(2) = 5$, $T(4) = 1$, and $T(6) = 1$. ■

Example 1.3.5

Functions and Relations on Sets of Strings

Let $A = \{a, b\}$ and let S be the set of all strings over A.

a. Define a relation L from S to $\mathbf{Z}^{nonneg}$ as follows: For every string s in S and for every nonnegative integer n,

$$(s, n) \in L \text{ means that the length of } s \text{ is } n.$$

Observe that L is a function because every string in S has one and only one length. Find $L(abaaba)$ and $L(bbb)$.

b. Define a relation C from S to S as follows: For all strings s and t in S,

$$(s, t) \in C \text{ means that } t = as,$$

where as is the string obtained by appending a on the left of the characters in s. (C is called **concatenation** by a on the left.) Observe that C is a function because every string in S consists entirely of a's and b's and adding an additional a on the left creates a new strong that also consists of a's and b's and thus is also in S. Find $C(abaaba)$ and $C(bbb)$.

Solution

a. $L(abaaba) = 6$ and $L(bbb) = 3$

b. $C(abaaba) = aabaaba$ and $C(bbb) = abbb$ ■

Function Machines

Another useful way to think of a function is as a machine. Suppose f is a function from X to Y and an input x of X is given. Imagine f to be a machine that processes x in a certain way to produce the output $f(x)$. This is illustrated in Figure 1.3.1.

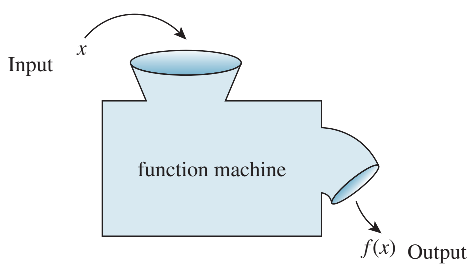


FIGURE 1.3.1

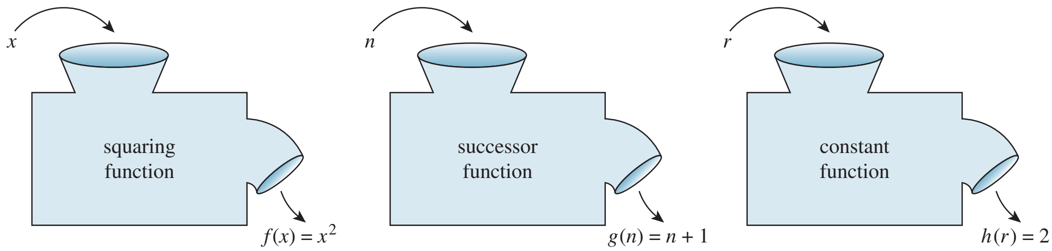
Example 1.3.6 Functions Defined by Formulas

The **squaring function** f from $\mathbf{R}$ to $\mathbf{R}$ is defined by the formula $f(x) = x^2$ for every real number x. This means that no matter what real number input is substituted for x, the output of f will be the square of that number. This idea can be represented by writing $f(\square) = \square^2$. In other words, f sends each real number x to x^2, or, symbolically, $f\colon x \to x^2$. Note that the variable x is a dummy variable; any other symbol could replace it, as long as the replacement is made everywhere the x appears.

The **successor function** g from $\mathbf{Z}$ to $\mathbf{Z}$ is defined by the formula $g(n) = n + 1$. Thus, no matter what integer is substituted for n, the output of g will be that number plus 1: $g(\square) = \square + 1$. In other words, g sends each integer n to $n + 1$, or, symbolically, $g\colon n \to n + 1$.

An example of a **constant function** is the function h from $\mathbf{Q}$ to $\mathbf{Z}$ defined by the formula $h(r) = 2$ for all rational numbers r. This function sends each rational number r to 2. In other words, no matter what the input, the output is always 2: $h(\square) = 2$ or $h\colon r \to 2$.

The functions f, g, and h are represented by the function machines in Figure 1.3.2.



(a) (b) (c)

FIGURE 1.3.2 ■

A function is an entity in its own right. It can be thought of as a certain relationship between sets or as an input/output machine that operates according to a certain rule. This is the reason why a function is generally denoted by a single symbol or string of symbols, such as f, G, of log, or sin.

A relation is a subset of a Cartesian product and a function is a special kind of relation. Specifically, if f and g are functions from a set A to a set B, then

$$f = \{(x, y) \in A \times B \mid y = f(x)\} \quad \text{and} \quad g = \{(x, y) \in A \times B \mid y = g(x)\}.$$

It follows that

$$\boldsymbol{f} \textbf{ equals } \boldsymbol{g}, \quad \text{written } \boldsymbol{f = g}, \quad \text{if, and only if,} \quad \boldsymbol{f(x) = g(x)} \text{ for all } x \text{ in } A.$$

Example 1.3.7 Equality of Functions

Define functions f and g from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas:

$$f(x) = |x| \quad \text{for every } x \in \mathbf{R}.$$
$$g(x) = \sqrt{x^2} \quad \text{for every } x \in \mathbf{R}.$$

Does $f = g$?

Solution

Yes. Because the absolute value of any real number equals the square root of its square, $|x| = \sqrt{x^2}$ for all $x \in \mathbf{R}$. Hence $f = g$. ■

TEST YOURSELF

1. Given sets A and B, a relation from A to B is ______.
2. A function F from A to B is a relation from A to B that satisfies the following two properties:
   a. for every element x of A, there is ______.
   b. for all elements x in A and y and z in B, if ______ then ______.
3. If F is a function from A to B and x is an element of A, then $F(x)$ is ______.

EXERCISE SET 1.3

1. Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a relation R from A to B as follows: For every $(x, y) \in A \times B$,
$$(x, y) \in R \quad \text{means that} \quad \frac{y}{x} \text{ is an integer.}$$
   a. Is $4\ R\ 6$? Is $4\ R\ 8$? Is $(3, 8) \in R$? Is $(2, 10) \in R$?
   b. Write R as a set of ordered pairs.
   c. Write the domain and co-domain of R.
   d. Draw an arrow diagram for R.
2. Let $C = D = \{-3, -2, -1, 1, 2, 3\}$ and define a relation S from C to D as follows: For every $(x, y) \in C \times D$,
$$(x, y) \in S \quad \text{means that} \quad \tfrac{1}{x} - \tfrac{1}{y} \text{ is an integer.}$$
   a. Is $2\ S\ 2$? Is $-1\ S\ -1$? Is $(3, 3) \in S$? Is $(3, -3) \in S$?
   b. Write S as a set of ordered pairs.
   c. Write the domain and co-domain of S.
   d. Draw an arrow diagram for S.
3. Let $E = \{1, 2, 3\}$ and $F = \{-2, -1, 0\}$ and define a relation T from E to F as follows: For every $(x, y) \in E \times F$,
$$(x, y) \in T \quad \text{means that} \quad \frac{x - y}{3} \text{ is an integer.}$$
   a. Is $3\ T\ 0$? Is $1\ T\ (-1)$? Is $(2, -1) \in T$? Is $(3, -2) \in T$?
   b. Write T as a set of ordered pairs.
   c. Write the domain and co-domain of T.
   d. Draw an arrow diagram for T.
4. Let $G = \{-2, 0, 2\}$ and $H = \{4, 6, 8\}$ and define a relation V from G to H as follows: For every $(x, y) \in G \times H$,
$$(x, y) \in V \quad \text{means that} \quad \frac{x - y}{4} \text{ is an integer.}$$
   a. Is $2\ V\ 6$? Is $(-2)\ V\ (8)$? Is $(0, 6) \in V$? Is $(2, 4) \in V$?
   b. Write V as a set of ordered pairs.
   c. Write the domain and co-domain of V.
   d. Draw an arrow diagram for V.
5. Define a relation S from $\mathbf{R}$ to $\mathbf{R}$ as follows: For every $(x, y) \in \mathbf{R} \times \mathbf{R}$,
$$(x, y) \in S \quad \text{means that} \quad x \geq y.$$
   a. Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $2\ S\ 3$? Is $(-1)\ S\ (-2)$?
   b. Draw the graph of S in the Cartesian plane.
6. Define a relation R from $\mathbf{R}$ to $\mathbf{R}$ as follows: For every $(x, y) \in \mathbf{R} \times \mathbf{R}$,
$$(x, y) \in R \quad \text{means that} \quad y = x^2.$$
   a. Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(-3)\ R\ 9$? Is $9\ R\ (-3)$?
   b. Draw the graph of R in the Cartesian plane.
7. Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define relations R, S, and T from A to B as follows: For every $(x, y) \in A \times B$:
$$(x, y) \in R \quad \text{means that} \quad x \geq y.$$
$$(x, y) \in S \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$
$$T = \{(4, 7), (6, 5), (6, 7)\}.$$
   a. Draw arrow diagrams for R, S, and T.
   b. Indicate whether any of the relations R, S, and T are functions.
8. Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations U, V, and W from A to B as follows: For every $(x, y) \in A \times B$:
$$(x, y) \in U \quad \text{means that} \quad y - x > 2.$$
$$(x, y) \in V \quad \text{means that} \quad y - 1 = \tfrac{x}{2}.$$
$$W = \{(2, 5), (4, 1), (2, 3)\}.$$
   a. Draw arrow diagrams for U, V, and W.
   b. Indicate whether any of the relations U, V, and W are functions.

9. **a.** Find all functions from {0, 1} to {1}.
b. Find two relations from {0, 1} to {1} that are not functions.

10. Find four relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

11. Let $A = \{0, 1, 2\}$ and let S be the set of all strings over A. Define a relation L from S to $\mathbf{Z}^{nonneg}$ as follows: For every string s in S and every nonnegative integer n,

$$(s, n) \in L \quad \text{means that} \quad \text{the length of } s \text{ is } n.$$

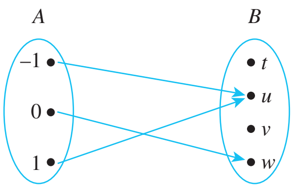
Then L is a function because every string in S has one and only one length. Find $L(0201)$ and $L(12)$.

12. Let $A = \{x, y\}$ and let S be the set of all strings over A. Define a relation C from S to S as follows: For all strings s and t in S,

$$(s, t) \in C \quad \text{means that} \quad t = ys.$$

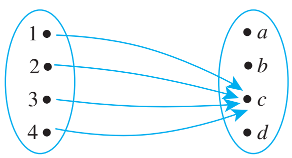
Then C is a function because every string in S consists entirely of x's and y's and adding an additional y on the left creates a single new string that consists of x's and y's and is, therefore, also in S. Find $C(x)$ and $C(yyxyx)$.

13. Let $A = \{-1, 0, 1\}$ and $B = \{t, u, v, w\}$. Define a function $F: A \to B$ by the following arrow diagram:



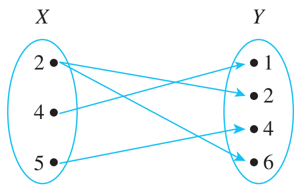
a. Write the domain and co-domain of F.
b. Find $F(-1)$, $F(0)$, and $F(1)$.

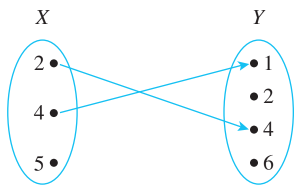
14. Let $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$. Define a function $G: C \to D$ by the following arrow diagram:

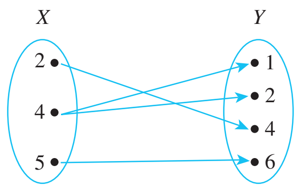


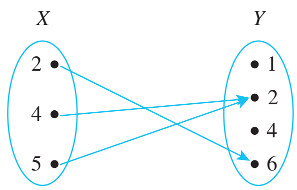
a. Write the domain and co-domain of G.
b. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

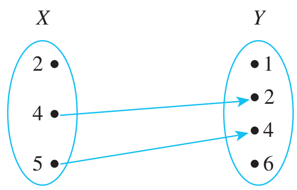
15. Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from X to Y?

a.


b.


c.


d.


e.


16. Let f be the squaring function defined in Example 1.3.6. Find $f(-1)$, $f(0)$, and $f\left(\frac{1}{2}\right)$.

17. Let g be the successor function defined in Example 1.3.6. Find $g(-1000)$, $g(0)$, and $g(999)$.

18. Let h be the constant function defined in Example 1.3.6. Find $h\left(-\frac{12}{5}\right)$, $h\left(\frac{0}{1}\right)$, and $h\left(\frac{9}{17}\right)$.

19. Define functions f and g from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For every $x \in \mathbf{R}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Does $f = g$? Explain.

20. Define functions H and K from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For every $x \in \mathbf{R}$,

$$H(x) = (x - 2)^2 \quad \text{and} \quad K(x) = (x - 1)(x - 3) + 1.$$

Does $H = K$? Explain.

ANSWERS FOR TEST YOURSELF

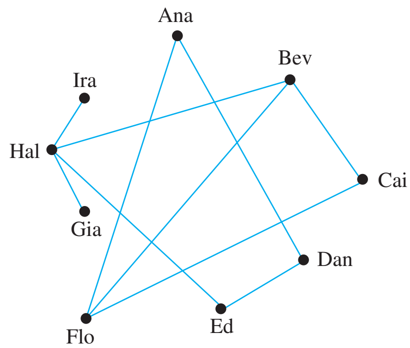
1. a subset of the Cartesian product $A \times B$ **2.** a. an element y of B such that $(x, y) \in F$ (i.e., such that x is related to y by F) b. $(x, y) \in F$ and $(x, z) \in F$; $y = z$ **3.** the unique element of B that is related to x by F

1.4 The Language of Graphs

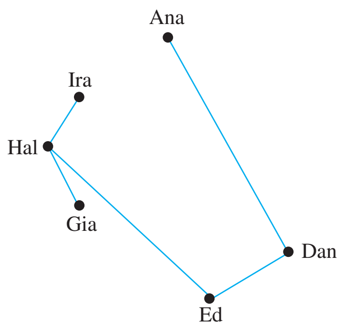
The whole of mathematics consists in the organization of a series of aids to the imagination in the process of reasoning. —Alfred North Whitehead, 1861–1947

Imagine an organization that wants to set up teams of three to work on some projects. In order to maximize the number of people on each team who had previous experience working together successfully, the director asked the members to provide names of their previous partners. This information is displayed below both in a table and in a diagram.

Name	Previous Partners
Ana	Dan, Flo
Bev	Cai, Flo, Hal
Cai	Bev, Flo
Dan	Ana, Ed
Ed	Dan, Hal
Flo	Cai, Bev, Ana
Gia	Hal
Hal	Gia, Ed, Bev, Ira
Ira	Hal



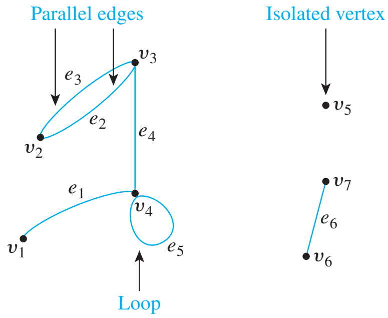
From the diagram, it is easy to see that Bev, Cai, and Flo are a group of three previous partners, and so it would be reasonable for them to form one of these teams. The drawing below shows the result when these three names are removed from the diagram.



This drawing shows that placing Hal on the same team as Ed would leave Gia and Ira on a team where they would not have a previous partner. However, if Hal is placed on a team with Gia and Ira, then the remaining team would consist of Ana, Dan, and Ed, and everyone on both teams would be working with a previous partner.

Drawings such as these are illustrations of a structure known as a *graph*. The dots are called *vertices* (plural of *vertex*) and the line segments joining vertices are called *edges*. As you can see from the first drawing, it is possible for two edges to cross at a point that is not a vertex. Note also that the type of graph described here is quite different from the "graph of an equation" or the "graph of a function."

In general, a graph consists of a set of vertices and a set of edges connecting various pairs of vertices. The edges may be straight or curved and should either connect one vertex to another or a vertex to itself, as shown below.



In this drawing, the vertices are labeled with v's and the edges with e's. When an edge connects a vertex to itself (as e_5 does), it is called a *loop*. When two edges connect the same pair of vertices (as e_2 and e_3 do), they are said to be *parallel*. It is quite possible for a vertex to be unconnected by an edge to any other vertex in the graph (as v_5 is), and in that case the vertex is said to be *isolated*. The formal definition of a graph follows.

Definition

A **graph** G consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**.

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

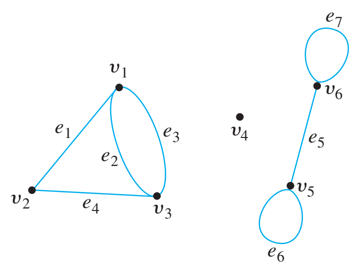
An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**.

Graphs have pictorial representations in which the vertices are represented by dots and the edges by line segments. A given pictorial representation uniquely determines a graph.

Example 1.4.1

Terminology

Consider the following graph:



a. Write the vertex set and the edge set, and give a table showing the edge-endpoint function.

b. Find all edges that are incident on v_1, all vertices that are adjacent to v_1, all edges that are adjacent to e_1, all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

Solution

a. vertex set $= \{v_1, v_2, v_3, v_4, v_5, v_6\}$
edge set $= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

Note The isolated vertex v_4 does not appear in the table. Although each edge of a graph must have either one or two endpoints, a vertex need not be an endpoint of an edge.

b. e_1, e_2, and e_3 are incident on v_1.
v_2 and v_3 are adjacent to v_1.
e_2, e_3, and e_4 are adjacent to e_1.
e_6 and e_7 are loops.
e_2 and e_3 are parallel.
v_5 and v_6 are adjacent to themselves.
v_4 is an isolated vertex. ■

Although a given pictorial representation uniquely determines a graph, a given graph may have more than one pictorial representation. Such things as the lengths or curvatures of the edges and the relative position of the vertices on the page may vary from one pictorial representation to another.

Example 1.4.2 Drawing More Than One Picture for a Graph

Consider the graph specified as follows:

vertex set $= \{v_1, v_2, v_3, v_4\}$
edge set $= \{e_1, e_2, e_3, e_4\}$
edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_2, v_4\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_3\}$