Blitzer

ALGEBRA AND TRIGONOMETRY

6th Edition

A Brief Guide to **Getting the Most** from this Book

1 Read the Book

Feature	Description	Benefit
Section-Opening Scenarios	Every section opens with a scenario presenting a unique application of algebra or trigonometry in your life outside the classroom.	Realizing that algebra and trigonometry are everywhere will help motivate your learning. **(See page 106.)**
Detailed Worked-Out Examples	Examples are clearly written and provide step-by-step solutions. No steps are omitted, and each step is thoroughly explained to the right of the mathematics.	The blue annotations will help you understand the solutions by providing the reason why every algebraic or trigonometric step is true. **(See page 674.)**
Applications Using Real-World Data	Interesting applications from nearly every discipline, supported by up-to-date real-world data, are included in every section.	Ever wondered how you'll use algebra and trigonometry? This feature will show you how algebra and trigonometry can solve real problems. **(See page 265.)**
Great Question!	Answers to students' questions offer suggestions for problem solving, point out common errors to avoid, and provide informal hints and suggestions.	By seeing common mistakes, you'll be able to avoid them. This feature should help you not to feel anxious or threatened when asking questions in class. **(See page 109.)**
Brief Reviews	NEW to this edition. Brief Reviews cover skills you already learned but may have forgotten.	Having these refresher boxes easily accessible will help ease anxiety about skills you may have forgotten. **(See page 478.)**
Achieving Success	NEW to this edition. Achieving Success boxes offer strategies for persistence and success in college mathematics courses.	Follow these suggestions to help achieve your full academic potential in college mathematics. **(See page 586.)**
Explanatory Voice Balloons	Voice balloons help to demystify algebra and trigonometry. They translate mathematical language into plain English, clarify problem-solving procedures, and present alternative ways of understanding.	Does math ever look foreign to you? This feature often translates math into everyday English. **(See page 201.)**
Learning Objectives	Every section begins with a list of objectives. Each objective is restated in the margin where the objective is covered.	The objectives focus your reading by emphasizing what is most important and where to find it. **(See page 633.)**
Technology	The screens displayed in the technology boxes show how graphing utilities verify and visualize algebraic and trigonometric results.	Even if you are not using a graphing utility in the course, this feature will help you understand different approaches to problem solving. **(See page 110.)**

2 Work the Problems

Feature	Description	Benefit
Check Point Examples	Each example is followed by a matched problem, called a Check Point, that offers you the opportunity to work a similar exercise. The answers to the Check Points are provided in the answer section.	You learn best by doing. You'll solidify your understanding of worked examples if you try a similar problem right away to be sure you understand what you've just read. **(See page 739.)**
Concept and Vocabulary Checks	These short-answer questions, mainly fill-in-the-blank and true/false items, assess your understanding of the definitions and concepts presented in each section.	It is difficult to learn algebra and trigonometry without knowing their special language. These exercises test your understanding of the vocabulary and concepts. **(See page 229.)**
Extensive and Varied Exercise Sets	An abundant collection of exercises is included in an Exercise Set at the end of each section. Exercises are organized within several categories. Your instructor will usually provide guidance on which exercises to work. The exercises in the first category, Practice Exercises, follow the same order as the section's worked examples.	The parallel order of the Practice Exercises lets you refer to the worked examples and use them as models for solving these problems. **(See page 406.)**
Practice Plus Problems	This category of exercises contains more challenging problems that often require you to combine several skills or concepts.	It is important to dig in and develop your problem-solving skills. Practice Plus Exercises provide you with ample opportunity to do so. **(See page 407.)**
Retaining the Concepts	NEW to this edition. Beginning with Chapter 2, each Exercise Set contains review exercises under the header "Retaining the Concepts."	These exercises improve your understanding of the topics and help maintain mastery of the material. **(See page 234.)**
Preview Problems	Each Exercise Set concludes with three problems to help you prepare for the next section.	These exercises let you review previously covered material that you'll need to be successful for the forthcoming section. Some of these problems will get you thinking about concepts you'll soon encounter. **(See page 660.)**

3 Review for Quizzes and Tests

Feature	Description	Benefit
Mid-Chapter Check Points	At approximately the midway point in the chapter, an integrated set of review exercises allows you to review the skills and concepts you learned separately over several sections.	By combining exercises from the first half of the chapter, the Mid-Chapter Check Points give a comprehensive review before you move on to the material in the remainder of the chapter. **(See page 776.)**
Chapter Review Grids	Each chapter contains a review chart that summarizes the definitions and concepts in every section of the chapter. Examples that illustrate these key concepts are also referenced in the chart.	Review this chart and you'll know the most important material in the chapter! **(See page 815.)**
Chapter Review Exercises	A comprehensive collection of review exercises for each of the chapter's sections follows the grid.	Practice makes perfect. These exercises contain the most significant problems for each of the chapter's sections. **(See page 209.)**
Chapter Tests	Each chapter contains a practice test with approximately 25 problems that cover the important concepts in the chapter. Take the practice test, check your answers, and then watch the Chapter Test Prep Videos to see worked-out solutions for any exercises you miss.	You can use the chapter test to determine whether you have mastered the material covered in the chapter. **(See page 213.)**
Chapter Test Prep Videos	These videos contain worked-out solutions to every exercise in each chapter test and can be found in MyMathLab and on YouTube.	The videos let you review any exercises you miss on the chapter test.
Objective Videos	NEW to this edition. These fresh, interactive videos walk you through the concepts from every objective of the text.	The videos provide you with active learning at your own pace.
Cumulative Review Exercises	Beginning with Chapter 2, each chapter concludes with a comprehensive collection of mixed cumulative review exercises. These exercises combine problems from previous chapters and the present chapter, providing an ongoing cumulative review.	Ever forget what you've learned? These exercises ensure that you are not forgetting anything as you move forward. **(See page 667.)**

ALGEBRA AND TRIGONOMETRY

6th EDITION

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Pearson

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Illustrations: Scientific Illustrators
Photo Research and Permission Clearance: Cenveo Publisher Services
Cover Design: Studio Montage
Cover Image: Ray_of_Light/Shutterstock

Library of Congress Cataloging-in-Publication Data

Names: Blitzer, Robert.
Title: Algebra and trigonometry / Robert Blitzer, Miami Dade College.
Description: Sixth edition. | Hoboken, NJ : Pearson Prentice Hall, [2018] |
Includes answers to selected exercises. | Includes subject index.
Identifiers: LCCN 2016042563 | ISBN 9780134463216
Subjects: LCSH: Algebra—Textbooks. | Trigonometry—Textbooks.
Classification: LCC QA152.3.B63 2018 | DDC 512/.13—dc23
LC record available at https://lccn.loc.gov/2016042563

Pearson

ISBN 13: 978-0-13-446321-6
ISBN 10: 0-13-446321-8

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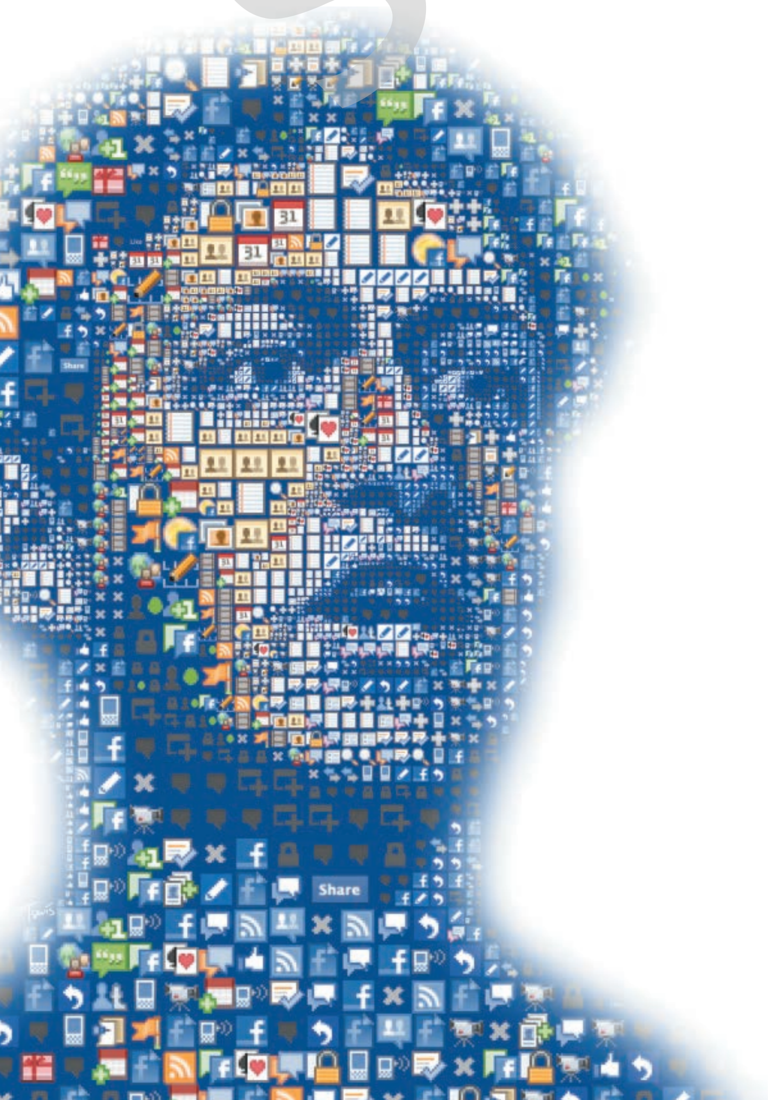

PREFACE

I've written ***Algebra and Trigonometry*, Sixth Edition**, to help diverse students, with different backgrounds and future goals, to succeed. The book has three fundamental goals:

1. To help students acquire a solid foundation in algebra and trigonometry, preparing them for other courses such as calculus, business calculus, and finite mathematics.
2. To show students how algebra and trigonometry can model and solve authentic real-world problems.
3. To enable students to develop problem-solving skills, while fostering critical thinking, within an interesting setting.

One major obstacle in the way of achieving these goals is the fact that very few students actually read their textbook. This has been a regular source of frustration for me and for my colleagues in the classroom. Anecdotal evidence gathered over years highlights two basic reasons that students do not take advantage of their textbook:

- "I'll never use this information."
- "I can't follow the explanations."

I've written every page of the Sixth Edition with the intent of eliminating these two objections. The ideas and tools I've used to do so are described for the student in "A Brief Guide to Getting the Most from This Book," which appears at the front of the book.

What's New in the Sixth Edition?

- **New Applications and Real-World Data.** The Sixth Edition contains 63 worked-out examples and exercises based on new data sets, and 36 examples and exercises based on data updated from the Fifth Edition. Many of the new applications involve topics relevant to college students, including student-loan debt (Chapter P, Mid-Chapter Check Point, Exercise 31), grade inflation (Exercise Set 1.2, Exercises 97–98), median earnings, by final degree earned (Exercise Set 1.3, Exercises 3–4), excuses for not meeting deadlines (Chapter 1 Summary, Exercise 36), political orientation of college freshmen (Chapter 2 Summary, Exercise 53), sleep hours of college students (Exercise Set 8.1, Exercise 74), and the number of hours college students study per week, by major (Exercise Set 8.2, Exercises 33–34).
- **Brief Reviews.** Beginning with Chapter 1, the Brief Review boxes that appear throughout the book summarize mathematical skills, many of which are course prerequisites, that students have learned, but which many students need to review. This feature appears whenever a particular skill is first needed and eliminates the need for you to reteach that skill. For more detail, students are referred to the appropriate section and objective in a previous chapter where the topic is fully developed.
- **Achieving Success.** The Achieving Success boxes, appearing at the end of many sections in Chapters 1 through 8, offer strategies for persistence and success in college mathematics courses.
- **Retaining the Concepts.** Beginning with Chapter 2, Section 2.1, each Exercise Set contains three or four review exercises under the header "Retaining the Concepts." These exercises are intended for students to review previously covered objectives in order to improve their understanding of the topics and to help maintain their mastery of the material. If students are not certain how to solve a review exercise, they can turn to the section and worked example given in parentheses at the end of each exercise. The Sixth Edition contains 216 new exercises in the "Retaining the Concepts" category.
- **New Blitzer Bonus Videos with Assessment.** Many of the Blitzer Bonus features throughout the textbook have been turned into animated videos that are built into the MyMathLab course. These videos help students make visual connections to algebra and trigonometry and the world around them. Assignable exercises have been created within the MyMathLab course to assess conceptual understanding and mastery. These videos and exercises can be turned into a media assignment within the Blitzer MyMathLab course.
- **Updated Learning Guide.** Organized by the textbook's learning objectives, this updated Learning Guide helps students make the most of their textbook for test preparation. Projects are now included to give students an opportunity to discover and reinforce the concepts in an active learning environment and are ideal for group work in class.
- **Updated Graphing Calculator Screens.** All screens have been updated using the TI-84 Plus C.

What Content and Organizational Changes Have Been Made to the Sixth Edition?

- **Section P.1 (Algebraic Expressions, Mathematical Models, and Real Numbers)** follows an example on the cost of attending college (Example 2) with a new Blitzer Bonus, "Is College Worthwhile?"
- **Section P.6 (Rational Expressions)** uses the least common denominator to combine rational expressions with different denominators, including expressions having no common factors in their denominators.
- **Section 1.1 (Graphing and Graphing Utilities)** contains a new example of a graph with more than one x-intercept (Example 5(d)).

- **Section 1.4 (Complex Numbers)** includes a new example on dividing complex numbers where the numerator is of the form bi (Example 3). (This is then followed by an example picked up from the Sixth Edition where the numerator is of the form $a + bi$.)
- **Section 1.5 (Quadratic Equations)** provides a step-by-step procedure for solving quadratic equations by completing the square. This procedure forms the framework for the solutions in Examples 4 and 5.
- **Section 1.5 (Quadratic Equations)** contains an example on the quadratic formula (Example 6) where the formula is used to solve a quadratic equation with rational solutions, an equation that students can also solve by factoring.
- **Section 1.5 (Quadratic Equations)** has a new application of the Pythagorean Theorem (Example 11) involving HDTV screens. The example is followed by a new Blitzer Bonus, "Screen Math."
- **Section 1.6 (Other Types of Equations)** includes an example on solving an equation quadratic in form (Example 8),

$$(x^2 - 5)^2 + 3(x^2 - 5) - 10 = 0,$$

  where u is a binomial ($u = x^2 - 5$).
- **Section 2.2 (More on Functions and Their Graphs)** contains a new discussion on graphs with three forms of symmetry (Examples 2 and 3) before presenting even and odd functions. A new example (Example 4) addresses identifying even or odd functions from graphs.
- **Section 2.3 (Linear Functions and Slope)** includes a new Blitzer Bonus, "Slope and Applauding Together."
- **Section 2.7 (Inverse Functions)** replaces an example on finding the inverse of $f(x) = \frac{5}{x} + 4$ with an example on finding the inverse of $f(x) = \frac{x+2}{x-3}$ (Example 4), a function with two occurrences of x.
- **Section 3.5 (Rational Functions and Their Graphs)** opens with a discussion of college students and video games. This is revisited in a new example (Example 9, "Putting the Video-Game Player Inside the Game") involving the Oculus Rift, a virtual reality headset that enables users to experience video games as immersive three-dimensional environments.
- **Section 5.1 (Angles and Radian Measure)** has new examples involving radians expressed in decimal form, including converting 2.3 radians to degrees (Example 3(d)) and finding a coterminal angle for a -10.3 angle (Example 7(d)). Additional Great Question! features provide hints for locating terminal sides of angles in standard position.
- **Section 5.2 (Right Triangle Trigonometry)** has a new Discovery feature on the use of parentheses when evaluating trigonometric functions with a graphing calculator, supported by new calculator screens throughout the section. A Great Question! has been added urging students not to become too calculator dependent.
- **Chapter 6** opens with a new discussion on trigonometric functions and music.
- **Section 8.1 (Systems of Linear Equations in Two Variables)** contains a new discussion on problems involving mixtures, important for many STEM students. A new example (Example 8) illustrates the procedure for solving a mixture problem.
- **Section 9.1 (Matrix Solutions to Linear Systems)** has a new opening example (Example 1) showing the details on how to write an augmented matrix.
- **Section 10.1 (The Ellipse)** includes a new example (Example 5) showing the details on graphing an ellipse centered at (h, k) by completing the square.
- **Section 10.3 (The Parabola)** adds a new objective, moved from Section 10.4 (Rotation of Axes), on identifying conics of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ without completing the square, supported by an example (Example 7).
- **Section 11.2 (Arithmetic Sequences)** contains a new example (Example 3) on writing the general term of an arithmetic sequence.
- **Section 11.7 (Probability)** uses the popular lottery games Powerball (Example 5) and Mega Millions (Exercises 27–30) as applications of probability and combinations.

What Familiar Features Have Been Retained in the Sixth Edition?

- **Learning Objectives.** Learning objectives, framed in the context of a student question (What am I supposed to learn?), are clearly stated at the beginning of each section. These objectives help students recognize and focus on the section's most important ideas. The objectives are restated in the margin at their point of use.
- **Chapter-Opening and Section-Opening Scenarios.** Every chapter and every section open with a scenario presenting a unique application of mathematics in students' lives outside the classroom. These scenarios are revisited in the course of the chapter or section in an example, discussion, or exercise.
- **Innovative Applications.** A wide variety of interesting applications, supported by up-to-date, real-world data, are included in every section.
- **Detailed Worked-Out Examples.** Each example is titled, making the purpose of the example clear. Examples are clearly written and provide students with detailed step-by-step solutions. No steps are omitted and each step is thoroughly explained to the right of the mathematics.

- **Explanatory Voice Balloons.** Voice balloons are used in a variety of ways to demystify mathematics. They translate algebraic and trigonometric ideas into everyday English, help clarify problem-solving procedures, present alternative ways of understanding concepts, and connect problem solving to concepts students have already learned.
- **Check Point Examples.** Each example is followed by a similar matched problem, called a Check Point, offering students the opportunity to test their understanding of the example by working a similar exercise. The answers to the Check Points are provided in the answer section.
- **Concept and Vocabulary Checks.** This feature offers short-answer exercises, mainly fill-in-the-blank and true/false items, that assess students' understanding of the definitions and concepts presented in each section. The Concept and Vocabulary Checks appear as separate features preceding the Exercise Sets.
- **Extensive and Varied Exercise Sets.** An abundant collection of exercises is included in an Exercise Set at the end of each section. Exercises are organized within nine category types: Practice Exercises, Practice Plus Exercises, Application Exercises, Explaining the Concepts, Technology Exercises, Critical Thinking Exercises, Group Exercises, Retaining the Concepts, and Preview Exercises. This format makes it easy to create well-rounded homework assignments. The order of the Practice Exercises is exactly the same as the order of the section's worked examples. This parallel order enables students to refer to the titled examples and their detailed explanations to achieve success working the Practice Exercises.
- **Practice Plus Problems.** This category of exercises contains more challenging practice problems that often require students to combine several skills or concepts. With an average of ten Practice Plus problems per Exercise Set, instructors are provided with the option of creating assignments that take Practice Exercises to a more challenging level.
- **Mid-Chapter Check Points.** At approximately the midway point in each chapter, an integrated set of Review Exercises allows students to review and assimilate the skills and concepts they learned separately over several sections.
- **Graphing and Functions.** Graphing is introduced in Chapter 1 and functions are introduced in Chapter 2, with an integrated graphing functional approach emphasized throughout the book. Graphs and functions that model data appear in nearly every section and Exercise Set. Examples and exercises use graphs of functions to explore relationships between data and to provide ways of visualizing a problem's solution. Because functions are the core of this course, students are repeatedly shown how functions relate to equations and graphs.
- **Integration of Technology Using Graphic and Numerical Approaches to Problems.** Side-by-side features in the technology boxes connect algebraic and trigonometric solutions to graphic and numerical approaches to problems. Although the use of graphing utilities is optional, students can use the explanatory voice balloons to understand different approaches to problems even if they are not using a graphing utility in the course.
- **Great Question!** This feature presents a variety of study tips in the context of students' questions. Answers to questions offer suggestions for problem solving, point out common errors to avoid, and provide informal hints and suggestions. As a secondary benefit, this feature should help students not to feel anxious or threatened when asking questions in class.
- **Chapter Summaries.** Each chapter contains a review chart that summarizes the definitions and concepts in every section of the chapter. Examples that illustrate these key concepts are also referenced in the chart.
- **End-of-Chapter Materials.** A comprehensive collection of Review Exercises for each of the chapter's sections follows the Summary. This is followed by a Chapter Test that enables students to test their understanding of the material covered in the chapter. Beginning with Chapter 2, each chapter concludes with a comprehensive collection of mixed Cumulative Review Exercises.
- **Blitzer Bonuses.** These enrichment essays provide historical, interdisciplinary, and otherwise interesting connections to the algebra and trigonometry under study, showing students that math is an interesting and dynamic discipline.
- **Discovery.** Discovery boxes, found throughout the text, encourage students to further explore algebraic and trigonometric concepts. These explorations are optional and their omission does not interfere with the continuity of the topic under consideration.

I hope that my passion for teaching, as well as my respect for the diversity of students I have taught and learned from over the years, is apparent throughout this new edition. By connecting algebra and trigonometry to the whole spectrum of learning, it is my intent to show students that their world is profoundly mathematical, and indeed, π is in the sky.

Robert Blitzer

Acknowledgments

An enormous benefit of authoring a successful series is the broad-based feedback I receive from the students, dedicated users, and reviewers. Every change to this edition is the result of their thoughtful comments and suggestions. I would like to express my appreciation to all the reviewers, whose collective insights form the backbone of this revision. In particular, I would like to thank the following people for reviewing *College Algebra*, *Algebra and Trigonometry*, *Precalculus*, and *Trigonometry*.

Karol Albus, South Plains College
Kayoko Yates Barnhill, Clark College
Timothy Beaver, Isothermal Community College
Jaromir Becan, University of Texas-San Antonio
Imad Benjelloun, Delaware Valley College
Lloyd Best, Pacific Union College
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Michael Schramm, Indian River State College

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Philip Van Veldhuizen, University of Nevada at Reno

Philip Veer, Johnson County Community College

Jeffrey Weaver, Baton Rouge Community College

Amanda Wheeler, Amarillo College

David White, The Victoria College

Tracy Wienckowski, University of Buffalo

Additional acknowledgments are extended to Dan Miller and Kelly Barber for preparing the solutions manuals; Brad Davis for preparing the answer section, serving as accuracy checker, and writing the new learning guide; the codeMantra formatting team for the book's brilliant paging; Brian Morris and Kevin Morris at Scientific Illustrators for superbly illustrating the book; Francesca Monaco, project manager; and Kathleen Manley, production editor, whose collective talents kept every aspect of this complex project moving through its many stages.

I would like to thank my editor at Pearson, Dawn Murrin, who, with the assistance of Joseph Colella, guided and coordinated the book from manuscript through production. Finally, thanks to Peggy Lucas and Claire Kozar for their innovative marketing efforts and to the entire Pearson sales force for their confidence and enthusiasm about the book.

Robert Blitzer

Get the Most Out of MyMathLab®

MyMathLab is the leading online homework, tutorial, and assessment program for teaching and learning mathematics, built around Pearson's best-selling content. MyMathLab helps students and instructors improve results; it provides engaging experiences and personalized learning for each student so learning can happen in any environment. Plus, it offers flexible and time-saving course management features to allow instructors to easily manage their classes while remaining in complete control, regardless of course format.

Preparedness

MyMathLab course solutions offer a complete College Algebra, Algebra & Trigonometry, or Precalculus course with integrated review of select topics from developmental algebra. These courses help remediate students "just-in-time" and help with student retention of important concepts, ultimately boosting student success.

- Students begin each chapter by completing a Skills Check assignment to pinpoint which developmental topics, if any, they need to review.
- Students who demonstrate mastery of the review topics will move straight into the Algebra & Trigonometry content.
- A personalized review homework assignment will provide extra support for the students who need it.
- Additional review materials (worksheets, videos, and more) are available in an Integrated Review section at the start of each chapter in MyMathLab.

MyMathLab with Integrated Review are appropriate for students who struggle with pre-requisite skills and for co-requisite course models. These Integrated Review MyMathLab courses are available for a variety of College Algebra, Algebra & Trigonometry and Precalculus programs, as well as a variety of other disciplines.

Used by more than 37 million students worldwide, MyMathLab delivers consistent, measurable gains in student learning outcomes, retention, and subsequent course success.

MyMathLab Online Course for *Algebra and Trigonometry* by Robert Blitzer

(access code required)

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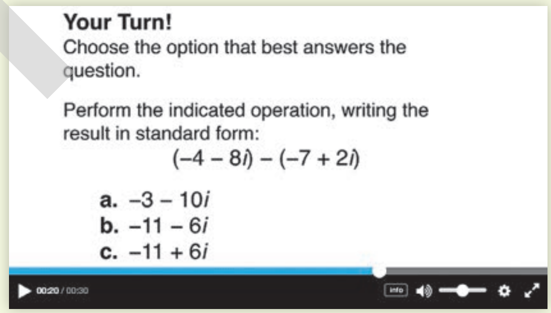
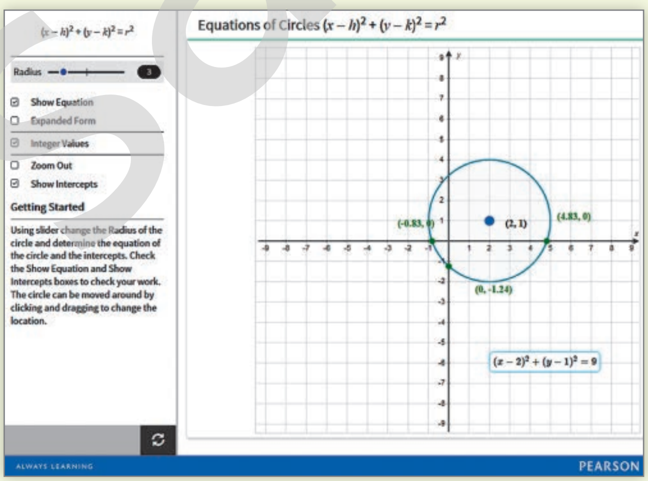
NEW! Video Program

A fresh, and all new, video program walks through the concepts from every objective of the text. Many videos provide an active learning environment where students try out their newly learned skill.





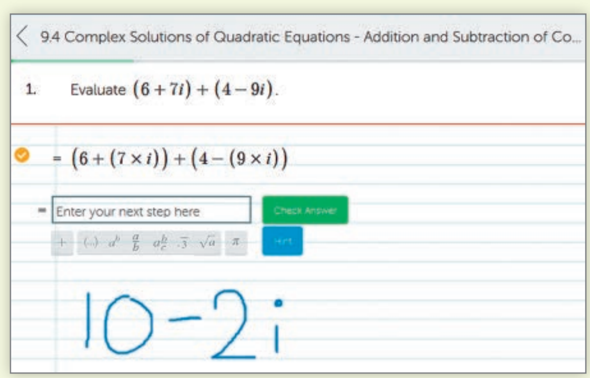
NEW! Guided Visualizations

These HTML-based, interactive figures help students visualize the concepts through directed explorations and purposeful manipulation. They encourage active learning, critical thinking, and conceptual learning. They are compatible with iPad and tablet devices.

The Guided Visualizations are located in the Multimedia Library and can be assigned as homework with correlating assessment exercises. Additional Exploratory Exercises are available to help students think more conceptually about the figures and provide an excellent framework for group projects or lecture discussion.

NEW! Workspace Assignments

Students can now show their work like never before! Workspace Assignments allow students to work through an exercise step-by-step, and show their mathematical reasoning as they progress. Students receive immediate feedback after they complete each step, and helpful hints and videos offer guidance when they need it. When accessed via a mobile device, Workspace exercises use handwriting recognition software that allows students to naturally write out their answers. Each student's work is automatically graded and captured in the MyMathLab gradebook so instructors can easily pinpoint exactly where they need to focus their instruction.



www.mymathlab.com

Resources for Success

Instructor Resources

Additional resources can be downloaded from **www.mymathlab.com** or **www.pearsonhighered.com** or hardcopy resources can be ordered from your sales representative.

Annotated Instructor's Edition

Shorter answers are on the page beside the exercises. Longer answers are in the back of the text.

Instructor's Solutions Manual

Fully worked solutions to all textbook exercises.

PowerPoint® Lecture Slides

Fully editable lecture slides that correlate to the textbook.

Mini Lecture Notes

Additional examples and helpful teaching tips for each section.

TestGen®

Enables instructors to build, edit, print, and administer tests using a computerized bank of algorithmic questions developed to cover all the objectives of the text.

Student Resources

Additional resources to help student success are available to be packaged with the Blitzer textbook and MyMathLab access code.

Objective Level Videos

An all new video program covers every objective of the text and is assignable in MyMathLab. Many videos provide an active learning environment where students try out their newly learned skill.

Chapter Test Prep Videos

Students can watch instructors work through step-by-step solutions to all the Chapter Test exercises from the textbook. These are available in MyMathLab and on YouTube.

Student Solutions Manual

Fully worked solutions to odd-numbered exercises and available to be packaged with the textbook.

Learning Guide

The note-taking guide begins each chapter with an engaging application, and provides additional examples and exercises for students to work through for a greater conceptual understanding and mastery of topics.
New to this edition: classroom projects are included for each chapter providing the opportunity for collaborative work. The Learning Guide is available in PDF and customizable Word file formats in MyMathLab. It can also be packaged with the textbook and MyMathLab access code.

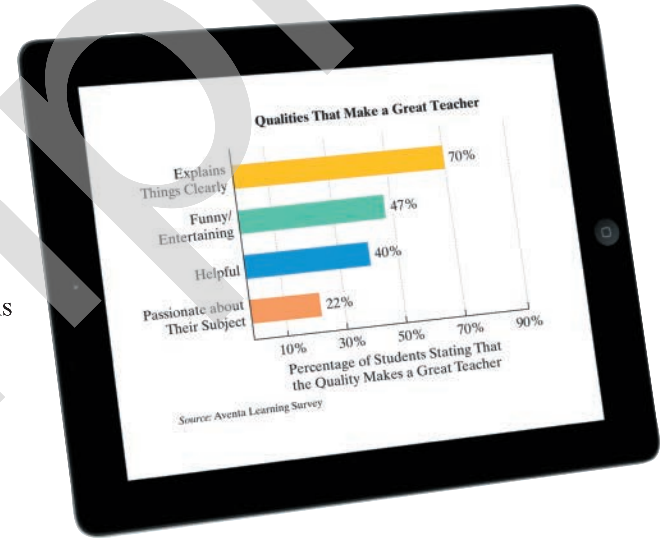
MathTalk Videos

Engaging videos connect mathematics to real-life events and interesting applications. These fun, instructional videos show students that math is relevant to their daily lives and are assignable in MyMathLab. Assignable exercises are available in MyMathLab for these videos to help students apply valuable information presented in the videos.

www.mymathlab.com

TO THE STUDENT

The bar graph shows some of the qualities that students say make a great teacher. It was my goal to incorporate each of these qualities throughout the pages of this book.



Explains Things Clearly

I understand that your primary purpose in reading *Algebra and Trigonometry* is to acquire a solid understanding of the required topics in your algebra and trigonometry course. In order to achieve this goal, I've carefully explained each topic. Important definitions and procedures are set off in boxes, and worked-out examples that present solutions in a step-by-step manner appear in every section. Each example is followed by a similar matched problem, called a Check Point, for you to try so that you can actively participate in the learning process as you read the book. (Answers to all Check Points appear in the back of the book.)

Funny & Entertaining

Who says that an algebra and trigonometry textbook can't be entertaining? From our unusual cover to the photos in the chapter and section openers, prepare to expect the unexpected. I hope some of the book's enrichment essays, called Blitzer Bonuses, will put a smile on your face from time to time.

Helpful

I designed the book's features to help you acquire knowledge of algebra and trigonometry, as well as to show you how algebra and trigonometry can solve authentic problems that apply to your life. These helpful features include:

- **Explanatory Voice Balloons:** Voice balloons are used in a variety of ways to make math less intimidating. They translate algebraic and trigonometric language into everyday English, help clarify problem-solving procedures, present alternative ways of understanding concepts, and connect new concepts to concepts you have already learned.
- **Great Question!:** The book's Great Question! boxes are based on questions students ask in class. The answers to these questions give suggestions for problem solving, point out common errors to avoid, and provide informal hints and suggestions.
- **Achieving Success:** The book's Achieving Success boxes give you helpful strategies for success in learning algebra and trigonometry, as well as suggestions that can be applied for achieving your full academic potential in future college coursework.
- **Chapter Summaries:** Each chapter contains a review chart that summarizes the definitions and concepts in every section of the chapter. Examples from the chapter that illustrate these key concepts are also referenced in the chart. Review these summaries and you'll know the most important material in the chapter!

Passionate about the Subject

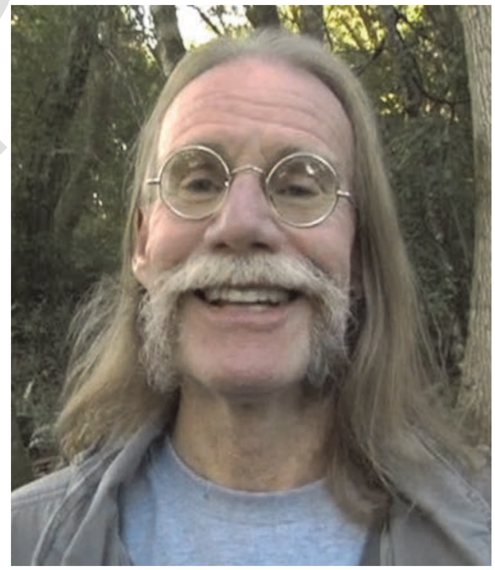
I passionately believe that no other discipline comes close to math in offering a more extensive set of tools for application and development of your mind. I wrote the book in Point Reyes National Seashore, 40 miles north of San Francisco. The park consists of 75,000 acres with miles of pristine surf-washed beaches, forested ridges, and bays bordered by white cliffs. It was my hope to convey the beauty and excitement of mathematics using nature's unspoiled beauty as a source of inspiration and creativity. Enjoy the pages that follow as you empower yourself with the algebra and trigonometry needed to succeed in college, your career, and your life.

Regards,

Bob

Robert Blitzer

ABOUT THE AUTHOR



Bob Blitzer is a native of Manhattan and received a Bachelor of Arts degree with dual majors in mathematics and psychology (minor: English literature) from the City College of New York. His unusual combination of academic interests led him toward a Master of Arts in mathematics from the University of Miami and a doctorate in behavioral sciences from Nova University. Bob's love for teaching mathematics was nourished for nearly 30 years at Miami Dade College, where he received numerous teaching awards, including Innovator of the Year from the League for Innovations in the Community College and an endowed chair based on excellence in the classroom. In addition to *Algebra and Trigonometry*, Bob has written textbooks covering developmental mathematics, introductory algebra, intermediate algebra, college algebra, trigonometry, precalculus, and liberal arts mathematics, all published by Pearson. When not secluded in his Northern California writer's cabin, Bob can be found hiking the beaches and trails of Point Reyes National Seashore and tending to the chores required by his beloved entourage of horses, chickens, and irritable roosters.

APPLICATIONS INDEX

Prerequisites: Fundamental Concepts of Algebra

CHAPTER P

What can algebra possibly have to tell me about

- the skyrocketing cost of a college education?
- student-loan debt?
- my workouts?
- the effects of alcohol?
- the meaning of the national debt that is nearly $19 trillion?
- time dilation on a futuristic high-speed journey to a nearby star?
- racial bias?
- ethnic diversity in the United States?
- the widening imbalance between numbers of women and men on college campuses?

This chapter reviews fundamental concepts of algebra that are prerequisites for the study of college algebra. Throughout the chapter, you will see how the special language of algebra describes your world.

HERE'S WHERE YOU'LL FIND THESE APPLICATIONS:

College costs: Section P.1, Example 2; Exercise Set P.1, Exercises 131–132

Student-loan debt: Mid-Chapter Check Point, Exercise 31

Workouts: Exercise Set P.1, Exercises 129–130

The effects of alcohol: Blitzer Bonus beginning on page 15

The national debt: Section P.2, Example 12

Time dilation: Blitzer Bonus on page 47

Racial bias: Exercise Set P.4, Exercises 91–92

U.S. ethnic diversity: Chapter P Review, Exercise 23

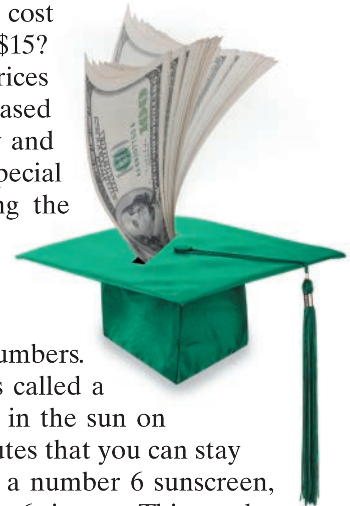
College gender imbalance: Chapter P Test, Exercise 32.

Section P.1 Algebraic Expressions, Mathematical Models, and Real Numbers

What am I supposed to learn?

After studying this section, you should be able to:

1. Evaluate algebraic expressions.
2. Use mathematical models.
3. Find the intersection of two sets.
4. Find the union of two sets.
5. Recognize subsets of the real numbers.
6. Use inequality symbols.
7. Evaluate absolute value.
8. Use absolute value to express distance.
9. Identify properties of the real numbers.
10. Simplify algebraic expressions.



How would your lifestyle change if a gallon of gas cost \$9.15? Or if the price of a staple such as milk was \$15? That's how much those products would cost if their prices had increased at the same rate college tuition has increased since 1980. (*Source*: Center for College Affordability and Productivity) In this section, you will learn how the special language of algebra describes your world, including the skyrocketing cost of a college education.

Algebraic Expressions

Algebra uses letters, such as x and y, to represent numbers. If a letter is used to represent various numbers, it is called a **variable**. For example, imagine that you are basking in the sun on the beach. We can let x represent the number of minutes that you can stay in the sun without burning with no sunscreen. With a number 6 sunscreen, exposure time without burning is six times as long, or 6 times x. This can be written $6 \cdot x$, but it is usually expressed as $6x$. Placing a number and a letter next to one another indicates multiplication.

Notice that $6x$ combines the number 6 and the variable x using the operation of multiplication. A combination of variables and numbers using the operations of addition, subtraction, multiplication, or division, as well as powers or roots, is called an **algebraic expression**. Here are some examples of algebraic expressions:

$$x + 6, \quad x - 6, \quad 6x, \quad \frac{x}{6}, \quad 3x + 5, \quad x^2 - 3, \quad \sqrt{x} + 7.$$

Many algebraic expressions involve *exponents*. For example, the algebraic expression

$$4x^2 + 330x + 3310$$

approximates the average cost of tuition and fees at public U.S. colleges for the school year ending x years after 2000. The expression x^2 means $x \cdot x$ and is read "x to the second power" or "x squared." The exponent, 2, indicates that the base, x, appears as a factor two times.

Exponential Notation

If n is a counting number (1, 2, 3, and so on),

Exponent or Power

$$b^n = \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{b \text{ appears as a factor } n \text{ times.}}.$$

Base

b^n is read "the nth power of b" or "b to the nth power." Thus, the nth power of b is defined as the product of n factors of b. The expression b^n is called an **exponential expression**. Furthermore, $b^1 = b$.

For example,

$$8^2 = 8 \cdot 8 = 64, \quad 5^3 = 5 \cdot 5 \cdot 5 = 125, \quad \text{and} \quad 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

1 Evaluate algebraic expressions.

Evaluating Algebraic Expressions

Evaluating an algebraic expression means to find the value of the expression for a given value of the variable.

Many algebraic expressions involve more than one operation. Evaluating an algebraic expression without a calculator involves carefully applying the following order of operations agreement:

The Order of Operations Agreement

1. Perform operations within the innermost parentheses and work outward. If the algebraic expression involves a fraction, treat the numerator and the denominator as if they were each enclosed in parentheses.
2. Evaluate all exponential expressions.
3. Perform multiplications and divisions **as they occur**, working **from left to right**.
4. Perform additions and subtractions **as they occur**, working **from left to right**.

EXAMPLE 1 Evaluating an Algebraic Expression

Evaluate $7 + 5(x - 4)^3$ for $x = 6$.

SOLUTION

$7 + 5(x - 4)^3 = 7 + 5(6 - 4)^3$	Replace x with 6.
$= 7 + 5(2)^3$	First work inside parentheses: $6 - 4 = 2$.
$= 7 + 5(8)$	Evaluate the exponential expression: $2^3 = 2 \cdot 2 \cdot 2 = 8$.
$= 7 + 40$	Multiply: $5(8) = 40$.
$= 47$	Add.

Check Point 1 Evaluate $8 + 6(x - 3)^2$ for $x = 13$.

2 Use mathematical models.

Formulas and Mathematical Models

An **equation** is formed when an equal sign is placed between two algebraic expressions. One aim of algebra is to provide a compact, symbolic description of the world. These descriptions involve the use of *formulas*. A **formula** is an equation that uses variables to express a relationship between two or more quantities.

Here are two examples of formulas related to heart rate and exercise.

Couch-Potato Exercise

$$H = \frac{1}{5}(220 - a)$$

Heart rate, in beats per minute, | is | $\frac{1}{5}$ of | the difference between 220 and your age.

Working It

$$H = \frac{9}{10}(220 - a)$$

Heart rate, in beats per minute, | is | $\frac{9}{10}$ of | the difference between 220 and your age.

The process of finding formulas to describe real-world phenomena is called **mathematical modeling**. Such formulas, together with the meaning assigned to the variables, are called **mathematical models**. We often say that these formulas model, or describe, the relationships among the variables.

EXAMPLE 2 Modeling the Cost of Attending a Public College

The bar graph in **Figure P.1** shows the average cost of tuition and fees for public four-year colleges, adjusted for inflation. The formula

$$T = 4x^2 + 330x + 3310$$

models the average cost of tuition and fees, T, for public U.S. colleges for the school year ending x years after 2000.

a. Use the formula to find the average cost of tuition and fees at public U.S. colleges for the school year ending in 2010.

b. By how much does the formula underestimate or overestimate the actual cost shown in **Figure P.1**?

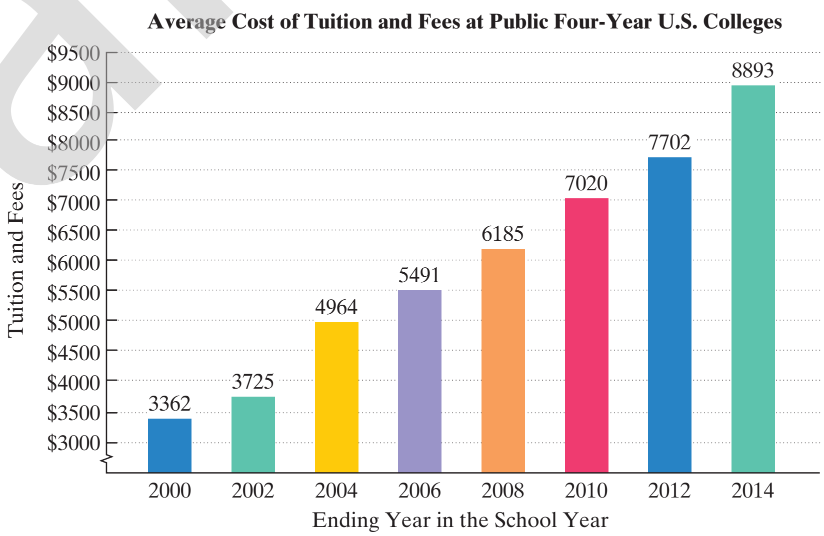


FIGURE P.1
Source: The College Board

SOLUTION

a. Because 2010 is 10 years after 2000, we substitute 10 for x in the given formula. Then we use the order of operations to find T, the average cost of tuition and fees for the school year ending in 2010.

$T = 4x^2 + 330x + 3310$	This is the given mathematical model.
$T = 4(10)^2 + 330(10) + 3310$	Replace each occurrence of x with 10.
$T = 4(100) + 330(10) + 3310$	Evaluate the exponential expression: $10^2 = 10 \cdot 10 = 100$.
$T = 400 + 3300 + 3310$	Multiply from left to right: $4(100) = 400$ and $330(10) = 3300$.
$T = 7010$	Add.

The formula indicates that for the school year ending in 2010, the average cost of tuition and fees at public U.S. colleges was \$7010.

b. **Figure P.1** shows that the average cost of tuition and fees for the school year ending in 2010 was \$7020.

The cost obtained from the formula, \$7010, underestimates the actual data value by \$7020 − \$7010, or by \$10. • • •

Blitzer Bonus || Is College Worthwhile?

"Questions have intensified about whether going to college is worthwhile," says *Education Pays*, released by the College Board Advocacy & Policy Center. "For the typical student, the investment pays off very well over the course of a lifetime, even considering the expense."

Among the findings in *Education Pays*:

- Mean (average) full-time earnings with a bachelor's degree in 2014 were \$62,504, which is \$27,768 more than high school graduates.
- Compared with a high school graduate, a four-year college graduate who enrolled in a public university at age 18 will break even by age 33. The college graduate will have earned enough by then to compensate for being out of the labor force for four years and for borrowing enough to pay tuition and fees, shown in **Figure P.1**.

Check Point 2

a. Use the formula $T = 4x^2 + 330x + 3310$, described in Example 2, to find the average cost of tuition and fees at public U.S. colleges for the school year ending in 2014.

b. By how much does the formula underestimate or overestimate the actual cost shown in **Figure P.1**?

Sometimes a mathematical model gives an estimate that is not a good approximation or is extended to include values of the variable that do not make sense. In these cases, we say that **model breakdown** has occurred. For example, it is not likely that the formula in Example 2 would give a good estimate of tuition and fees in 2050 because it is too far in the future. Thus, model breakdown would occur.

Sets

Before we describe the set of real numbers, let's be sure you are familiar with some basic ideas about sets. A **set** is a collection of objects whose contents can be clearly determined. The objects in a set are called the **elements** of the set. For example, the set of numbers used for counting can be represented by

$$\{1, 2, 3, 4, 5, \ldots\}.$$

GREAT QUESTION!

Can I use symbols other than braces when writing sets using the roster method?

No. Grouping symbols such as parentheses, ( ), and square brackets, [ ], are not used to represent sets in the roster method. Furthermore, only commas are used to separate the elements of a set. Separators such as colons or semicolons are not used.

The braces, { }, indicate that we are representing a set. This form of representation, called the **roster method**, uses commas to separate the elements of the set. The symbol consisting of three dots after the 5, called an *ellipsis*, indicates that there is no final element and that the listing goes on forever.

A set can also be written in **set-builder notation**. In this notation, the elements of the set are described but not listed. Here is an example:

$$\{\underbrace{x}_{\text{The set of all } x}\ \underbrace{|}_{\text{such that}}\ \underbrace{x \text{ is a counting number less than } 6}_{x \text{ is a counting number less than 6.}}\}.$$

The same set written using the roster method is

$$\{1, 2, 3, 4, 5\}.$$

3 Find the intersection of two sets.

If A and B are sets, we can form a new set consisting of all elements that are in both A and B. This set is called the *intersection* of the two sets.

Definition of the Intersection of Sets

The **intersection** of sets A and B, written $A \cap B$, is the set of elements common to both set A **and** set B. This definition can be expressed in set-builder notation as follows:

$$A \cap B = \{x | x \text{ is an element of } A \text{ AND } x \text{ is an element of } B\}.$$

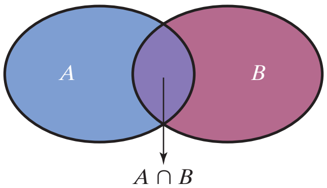


FIGURE P.2 Picturing the intersection of two sets

Figure P.2 shows a useful way of picturing the intersection of sets A and B. The figure indicates that $A \cap B$ contains those elements that belong to both A and B at the same time.

EXAMPLE 3 Finding the Intersection of Two Sets

Find the intersection: $\{7, 8, 9, 10, 11\} \cap \{6, 8, 10, 12\}$.

SOLUTION

The elements common to $\{7, 8, 9, 10, 11\}$ and $\{6, 8, 10, 12\}$ are 8 and 10. Thus,

$$\{7, 8, 9, 10, 11\} \cap \{6, 8, 10, 12\} = \{8, 10\}.$$

Check Point 3 Find the intersection: $\{3, 4, 5, 6, 7\} \cap \{3, 7, 8, 9\}$.

If a set has no elements, it is called the **empty set**, or the **null set**, and is represented by the symbol $\varnothing$ (the Greek letter phi). Here is an example that shows how the empty set can result when finding the intersection of two sets:

$$\{2, 4, 6\} \cap \{3, 5, 7\} = \varnothing.$$

These sets have no common elements.

Their intersection has no elements and is the empty set.

4 Find the union of two sets.

Another set that we can form from sets A and B consists of elements that are in A or B or in both sets. This set is called the *union* of the two sets.

Definition of the Union of Sets

The **union** of sets A and B, written $A \cup B$, is the set of elements that are members of set A **or** of set B or of both sets. This definition can be expressed in set-builder notation as follows:

$$A \cup B = \{x \mid x \text{ is an element of } A \text{ OR } x \text{ is an element of } B\}.$$

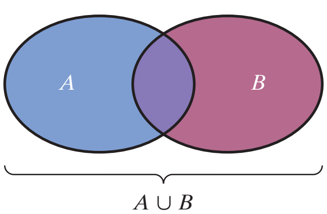


FIGURE P.3 Picturing the union of two sets

Figure P.3 shows a useful way of picturing the union of sets A and B. The figure indicates that $A \cup B$ is formed by joining the sets together.

We can find the union of set A and set B by listing the elements of set A. Then we include any elements of set B that have not already been listed. Enclose all elements that are listed with braces. This shows that the union of two sets is also a set.

EXAMPLE 4 Finding the Union of Two Sets

Find the union: $\{7, 8, 9, 10, 11\} \cup \{6, 8, 10, 12\}$.

SOLUTION

To find $\{7, 8, 9, 10, 11\} \cup \{6, 8, 10, 12\}$, start by listing all the elements from the first set, namely, 7, 8, 9, 10, and 11. Now list all the elements from the second set that are not in the first set, namely, 6 and 12. The union is the set consisting of all these elements. Thus,

$$\{7, 8, 9, 10, 11\} \cup \{6, 8, 10, 12\} = \{6, 7, 8, 9, 10, 11, 12\}.$$

Although 8 and 10 appear in both sets,

do not list 8 and 10 twice.

Check Point 4 Find the union: $\{3, 4, 5, 6, 7\} \cup \{3, 7, 8, 9\}$.

GREAT QUESTION!

How can I use the words *union* and *intersection* to help me distinguish between these two operations?

Union, as in a marriage union, suggests joining things, or uniting them. Intersection, as in the intersection of two crossing streets, brings to mind the area common to both, suggesting things that overlap.

5 Recognize subsets of the real numbers.

The Set of Real Numbers

The sets that make up the real numbers are summarized in **Table P.1**. We refer to these sets as **subsets** of the real numbers, meaning that all elements in each subset are also elements in the set of real numbers.

Table P.1 Important Subsets of the Real Numbers

Name/Symbol	Description	Examples
Natural numbers $\mathbb{N}$	$\{1, 2, 3, 4, 5, \ldots\}$ These are the numbers that we use for counting.	$2, 3, 5, 17$
Whole numbers $\mathbb{W}$	$\{0, 1, 2, 3, 4, 5, \ldots\}$ The set of whole numbers includes 0 and the natural numbers.	$0, 2, 3, 5, 17$
Integers $\mathbb{Z}$	$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$ The set of integers includes the negatives of the natural numbers and the whole numbers.	$-17, -5, -3, -2, 0, 2, 3, 5, 17$
Rational numbers $\mathbb{Q}$	$\left\{\frac{a}{b} \,\middle\vert\, a \text{ and } b \text{ are integers and } b \neq 0\right\}$ This means that b is not equal to zero. The set of rational numbers is the set of all numbers that can be expressed as a quotient of two integers, with the denominator not 0. Rational numbers can be expressed as terminating or repeating decimals.	$-17 = \frac{-17}{1}, -5 = \frac{-5}{1}, -3, -2,$ $0, 2, 3, 5, 17,$ $\frac{2}{5} = 0.4,$ $\frac{-2}{3} = -0.6666\ldots = -0.\overline{6}$
Irrational numbers $\mathbb{I}$	The set of irrational numbers is the set of all numbers whose decimal representations are neither terminating nor repeating. Irrational numbers cannot be expressed as a quotient of integers.	$\sqrt{2} \approx 1.414214$ $-\sqrt{3} \approx -1.73205$ $\pi \approx 3.142$ $-\frac{\pi}{2} \approx -1.571$

Notice the use of the symbol $\approx$ in the examples of irrational numbers. The symbol means "is approximately equal to." Thus,

$$\sqrt{2} \approx 1.414214.$$

TECHNOLOGY

A calculator with a square root key gives a decimal approximation for $\sqrt{2}$, not the exact value.

We can verify that this is only an approximation by multiplying 1.414214 by itself. The product is very close to, but not exactly, 2:

$$1.414214 \times 1.414214 = 2.000001237796.$$

Not all square roots are irrational. For example, $\sqrt{25} = 5$ because $5^2 = 5 \cdot 5 = 25$. Thus, $\sqrt{25}$ is a natural number, a whole number, an integer, and a rational number $\left(\sqrt{25} = \frac{5}{1}\right)$.

The set of *real numbers* is formed by taking the union of the sets of rational numbers and irrational numbers. Thus, every real number is either rational or irrational, as shown in **Figure P.4**.

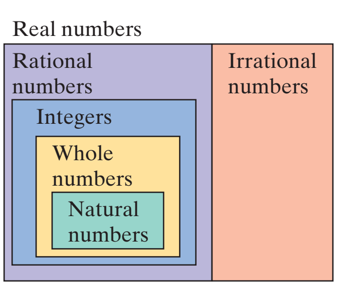


FIGURE P.4 Every real number is either rational or irrational.

Real Numbers

The set of **real numbers** is the set of numbers that are either rational or irrational:

$$\{x \mid x \text{ is rational or } x \text{ is irrational}\}.$$

The symbol $\mathbb{R}$ is used to represent the set of real numbers. Thus,

$$\mathbb{R} = \{x \mid x \text{ is rational}\} \cup \{x \mid x \text{ is irrational}\}.$$

EXAMPLE 5 Recognizing Subsets of the Real Numbers

Consider the following set of numbers:

$$\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}.$$

List the numbers in the set that are

a. natural numbers. **b.** whole numbers. **c.** integers.
d. rational numbers. **e.** irrational numbers. **f.** real numbers.

SOLUTION

a. Natural numbers: The natural numbers are the numbers used for counting. The only natural number in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ is $\sqrt{81}$ because $\sqrt{81} = 9$. (9 multiplied by itself, or 9^2, is 81.)

b. Whole numbers: The whole numbers consist of the natural numbers and 0. The elements of the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ that are whole numbers are 0 and $\sqrt{81}$.

c. Integers: The integers consist of the natural numbers, 0, and the negatives of the natural numbers. The elements of the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ that are integers are $\sqrt{81}$, 0, and -7.

d. Rational numbers: All numbers in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ that can be expressed as the quotient of integers are rational numbers. These include $-7\left(-7 = \frac{-7}{1}\right)$, $-\frac{3}{4}$, $0\left(0 = \frac{0}{1}\right)$, and $\sqrt{81}\left(\sqrt{81} = \frac{9}{1}\right)$. Furthermore, all numbers in the set that are terminating or repeating decimals are also rational numbers. These include $0.\overline{6}$ and 7.3.

e. Irrational numbers: The irrational numbers in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ are $\sqrt{5}\left(\sqrt{5} \approx 2.236\right)$ and $\pi(\pi \approx 3.14)$. Both $\sqrt{5}$ and π are only approximately equal to 2.236 and 3.14, respectively. In decimal form, $\sqrt{5}$ and π neither terminate nor have blocks of repeating digits.

f. Real numbers: All the numbers in the given set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ are real numbers.

Check Point 5 Consider the following set of numbers:

$$\left\{-9, -1.3, 0, 0.\overline{3}, \frac{\pi}{2}, \sqrt{9}, \sqrt{10}\right\}.$$

List the numbers in the set that are

a. natural numbers **b.** whole numbers **c.** integers.
d. rational numbers **e.** irrational numbers **f.** real numbers.

The Real Number Line

The **real number line** is a graph used to represent the set of real numbers. An arbitrary point, called the **origin**, is labeled 0. Select a point to the right of 0 and label it 1. The distance from 0 to 1 is called the **unit distance**. Numbers to the right of the origin are **positive** and numbers to the left of the origin are **negative**. The real number line is shown in **Figure P.5**.

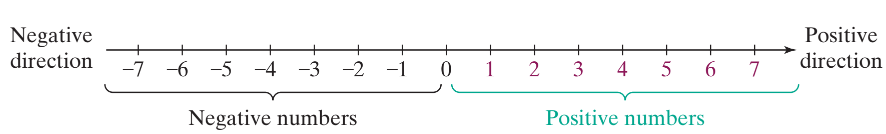
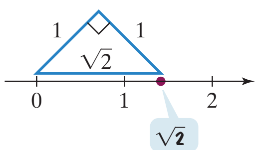


FIGURE P.5 The real number line

GREAT QUESTION!

How did you locate $\sqrt{2}$ as a precise point on the number line in Figure P.6?

We used a right triangle with two legs of length 1. The remaining side has a length measuring $\sqrt{2}$.



We'll have lots more to say about right triangles later in the book.

Real numbers are **graphed** on a number line by placing a dot at the correct location for each number. The integers are easiest to locate. In **Figure P.6**, we've graphed six rational numbers and three irrational numbers on a real number line.

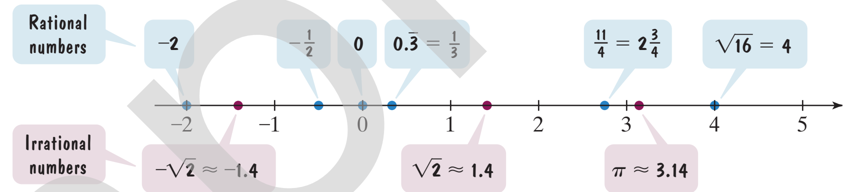


FIGURE P.6 Graphing numbers on a real number line

Every real number corresponds to a point on the number line and every point on the number line corresponds to a real number. We say that there is a **one-to-one correspondence** between all the real numbers and all points on a real number line.

6 Use inequality symbols.

Ordering the Real Numbers

On the real number line, the real numbers increase from left to right. The lesser of two real numbers is the one farther to the left on a number line. The greater of two real numbers is the one farther to the right on a number line.

Look at the number line in **Figure P.7**. The integers -4 and -1 are graphed.

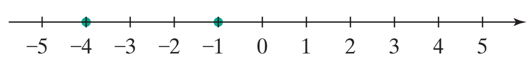


FIGURE P.7

Observe that -4 is to the left of -1 on the number line. This means that -4 is less than -1.

$$-4 < -1$$

-4 is less than -1 because -4 is to the left of -1 on the number line.

In **Figure P.7**, we can also observe that -1 is to the right of -4 on the number line. This means that -1 is greater than -4.

$$-1 > -4$$

-1 is greater than -4 because -1 is to the right of -4 on the number line.

The symbols $<$ and $>$ are called **inequality symbols**. These symbols always point to the lesser of the two real numbers when the inequality statement is true.

-4 is less than -1. $\quad -4 < -1 \quad$ *The symbol points to -4, the lesser number.*

-1 is greater than -4. $\quad -1 > -4 \quad$ *The symbol still points to -4, the lesser number.*

The symbols $<$ and $>$ may be combined with an equal sign, as shown in the following table:

Symbols	Meaning	Examples	Explanation
$a \le b$	a is less than or equal to b.	$2 \le 9$ $9 \le 9$	Because $2 < 9$ Because $9 = 9$
$b \ge a$	b is greater than or equal to a.	$9 \ge 2$ $2 \ge 2$	Because $9 > 2$ Because $2 = 2$

This inequality is true if either the $<$ part or the $=$ part is true.

This inequality is true if either the $>$ part or the $=$ part is true.

7 Evaluate absolute value.

Absolute Value

The **absolute value** of a real number a, denoted by $|a|$, is the distance from 0 to a on the number line. **This distance is always taken to be nonnegative.** For example, the real number line in **Figure P.8** shows that

$$|-3| = 3 \quad \text{and} \quad |5| = 5.$$

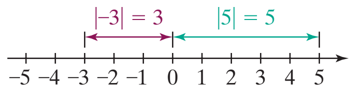


FIGURE P.8 Absolute value as the distance from 0

The absolute value of -3 is 3 because -3 is 3 units from 0 on the number line. The absolute value of 5 is 5 because 5 is 5 units from 0 on the number line. The absolute value of a positive real number or 0 is the number itself. The absolute value of a negative real number, such as -3, is the number without the negative sign.

We can define the absolute value of the real number x without referring to a number line. The algebraic definition of the absolute value of x is given as follows:

Definition of Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

If x is nonnegative (that is, $x \geq 0$), the absolute value of x is the number itself. For example,

$$|5| = 5 \qquad |\pi| = \pi \qquad \left|\frac{1}{3}\right| = \frac{1}{3} \qquad |0| = 0.$$

Zero is the only number whose absolute value is 0.

If x is a negative number (that is, $x < 0$), the absolute value of x is the opposite of x. This makes the absolute value positive. For example,

$$|-3| = -(-3) = 3 \qquad |-\pi| = -(-\pi) = \pi \qquad \left|-\frac{1}{3}\right| = -\left(-\frac{1}{3}\right) = \frac{1}{3}.$$

This middle step is usually omitted.

Observe that **the absolute value of any nonzero number is always positive**.

EXAMPLE 6 Evaluating Absolute Value

Rewrite each expression without absolute value bars:

a. $|\sqrt{3} - 1|$ **b.** $|2 - \pi|$ **c.** $\dfrac{|x|}{x}$ if $x < 0$.

SOLUTION

a. Because $\sqrt{3} \approx 1.7$, the number inside the absolute value bars, $\sqrt{3} - 1$, is positive. The absolute value of a positive number is the number itself. Thus,

$$|\sqrt{3} - 1| = \sqrt{3} - 1.$$

b. Because $\pi \approx 3.14$, the number inside the absolute value bars, $2 - \pi$, is negative. The absolute value of x when $x < 0$ is $-x$. Thus,

$$|2 - \pi| = -(2 - \pi) = \pi - 2.$$

c. If $x < 0$, then $|x| = -x$. Thus,

$$\frac{|x|}{x} = \frac{-x}{x} = -1.$$

Check Point 6 Rewrite each expression without absolute value bars:

a. $|1 - \sqrt{2}|$ **b.** $|\pi - 3|$ **c.** $\dfrac{|x|}{x}$ if $x > 0$.

Listed below are several basic properties of absolute value. Each of these properties can be derived from the definition of absolute value.

DISCOVERY

Verify the triangle inequality if $a = 4$ and $b = 5$. Verify the triangle inequality if $a = 4$ and $b = -5$.

When does equality occur in the triangle inequality and when does inequality occur? Verify your observation with additional number pairs.

Properties of Absolute Value

For all real numbers a and b,

1. $|a| \geq 0$ **2.** $|-a| = |a|$ **3.** $a \leq |a|$

4. $|ab| = |a||b|$ **5.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, $b \neq 0$

6. $|a + b| \leq |a| + |b|$ (called the triangle inequality)

8 Use absolute value to express distance.

Distance between Points on a Real Number Line

Absolute value is used to find the distance between two points on a real number line. If a and b are any real numbers, the **distance between *a* and *b*** is the absolute value of their difference. For example, the distance between 4 and 10 is 6. Using absolute value, we find this distance in one of two ways:

$$|10 - 4| = |6| = 6 \quad \text{or} \quad |4 - 10| = |-6| = 6.$$

The distance between 4 and 10 on the real number line is 6.

Notice that we obtain the same distance regardless of the order in which we subtract.

Distance between Two Points on the Real Number Line

If a and b are any two points on a real number line, then the distance between a and b is given by

$$|a - b| \quad \text{or} \quad |b - a|,$$

where $|a - b| = |b - a|$.

EXAMPLE 7 Distance between Two Points on a Number Line

Find the distance between -5 and 3 on the real number line.

SOLUTION

Because the distance between a and b is given by $|a - b|$, the distance between -5 and 3 is

$$|-5 - 3| = |-8| = 8.$$

$a = -5$ $b = 3$

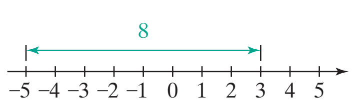


FIGURE P.9 The distance between -5 and 3 is 8.

Figure P.9 verifies that there are 8 units between -5 and 3 on the real number line. We obtain the same distance if we reverse the order of the subtraction:

$$|3 - (-5)| = |8| = 8.$$

Check Point 7 Find the distance between -4 and 5 on the real number line.

9 Identify properties of the real numbers.

Properties of Real Numbers and Algebraic Expressions

When you use your calculator to add two real numbers, you can enter them in any order. The fact that two real numbers can be added in any order is called the **commutative property of addition**. You probably use this property, as well as other properties of real numbers listed in **Table P.2**, without giving it much thought. The properties of the real numbers are especially useful when working with algebraic expressions. For each property listed in **Table P.2**, a, b, and c represent real numbers, variables, or algebraic expressions.

Table P.2 Properties of the Real Numbers

Name	Meaning	Examples
Commutative Property of Addition	Changing order when adding does not affect the sum. $a + b = b + a$	• $13 + 7 = 7 + 13$ • $13x + 7 = 7 + 13x$
Commutative Property of Multiplication	Changing order when multiplying does not affect the product. $ab = ba$	• $\sqrt{2} \cdot \sqrt{5} = \sqrt{5} \cdot \sqrt{2}$ • $x \cdot 6 = 6x$
Associative Property of Addition	Changing grouping when adding does not affect the sum. $(a + b) + c = a + (b + c)$	• $3 + (8 + x) = (3 + 8) + x$ $= 11 + x$
Associative Property of Multiplication	Changing grouping when multiplying does not affect the product. $(ab)c = a(bc)$	• $-2(3x) = (-2 \cdot 3)x = -6x$
Distributive Property of Multiplication over Addition	Multiplication distributes over addition. $a \cdot (b + c) = a \cdot b + a \cdot c$	• $7(4 + \sqrt{3}) = 7 \cdot 4 + 7 \cdot \sqrt{3}$ $= 28 + 7\sqrt{3}$ • $5(3x + 7) = 5 \cdot 3x + 5 \cdot 7$ $= 15x + 35$
Identity Property of Addition	Zero can be deleted from a sum. $a + 0 = a$ $0 + a = a$	• $\sqrt{3} + 0 = \sqrt{3}$ • $0 + 6x = 6x$
Identity Property of Multiplication	One can be deleted from a product. $a \cdot 1 = a$ $1 \cdot a = a$	• $1 \cdot \pi = \pi$ • $13x \cdot 1 = 13x$
Inverse Property of Addition	The sum of a real number and its additive inverse gives 0, the additive identity. $a + (-a) = 0$ $(-a) + a = 0$	• $\sqrt{5} + (-\sqrt{5}) = 0$ • $-\pi + \pi = 0$ • $6x + (-6x) = 0$ • $(-4y) + 4y = 0$
Inverse Property of Multiplication	The product of a nonzero real number and its multiplicative inverse gives 1, the multiplicative identity. $a \cdot \frac{1}{a} = 1, \quad a \neq 0$ $\frac{1}{a} \cdot a = 1, \quad a \neq 0$	• $7 \cdot \frac{1}{7} = 1$ • $\left(\frac{1}{x-3}\right)(x - 3) = 1, \quad x \neq 3$

The properties of the real numbers in **Table P.2** apply to the operations of addition and multiplication. Subtraction and division are defined in terms of addition and multiplication.

GREAT QUESTION!

Do the commutative and associative properties work for subtraction and division?

No. Subtraction and division are not commutative operations.

$$a - b \neq b - a \qquad \frac{a}{b} \neq \frac{b}{a}$$

Furthermore, subtraction and division are not associative operations.

$$(a - b) - c \neq a - (b - c)$$
$$(a \div b) \div c \neq a \div (b \div c)$$

Verify each of these four statements using $a = 10$, $b = 5$, and $c = 2$.

Definitions of Subtraction and Division

Let a and b represent real numbers.

Subtraction: $a - b = a + (-b)$

We call $-b$ the **additive inverse** or **opposite** of b.

Division: $a \div b = a \cdot \frac{1}{b}$, where $b \neq 0$

We call $\frac{1}{b}$ the **multiplicative inverse** or **reciprocal** of b. The quotient of a and b, $a \div b$, can be written in the form $\frac{a}{b}$, where a is the **numerator** and b is the **denominator** of the fraction.

Because subtraction is defined in terms of adding an inverse, the distributive property can be applied to subtraction:

$$a(b - c) = ab - ac$$
$$(b - c)a = ba - ca.$$

For example,

$$4(2x - 5) = 4 \cdot 2x - 4 \cdot 5 = 8x - 20.$$

10 Simplify algebraic expressions.

Simplifying Algebraic Expressions

The **terms** of an algebraic expression are those parts that are separated by addition. For example, consider the algebraic expression

$$7x - 9y + z - 3,$$

which can be expressed as

$$7x + (-9y) + z + (-3).$$

This expression contains four terms, namely, $7x$, $-9y$, z, and -3.

The numerical part of a term is called its **coefficient**. In the term $7x$, the 7 is the coefficient. If a term containing one or more variables is written without a coefficient, the coefficient is understood to be 1. Thus, z means $1z$. If a term is a constant, its coefficient is that constant. Thus, the coefficient of the constant term -3 is -3.

$$7x + (-9y) + z + (-3)$$

Coefficient is 7.

Coefficient is −9.

Coefficient is 1; z means $1z$.

Coefficient is −3.

The parts of each term that are multiplied are called the **factors** of the term. The **factors** of the term $7x$ are 7 and x.

Like terms are terms that have exactly the same variable factors. For example, $3x$ and $7x$ are like terms. The distributive property in the form

$$ba + ca = (b + c)a$$

enables us to add or subtract like terms. For example,

$$3x + 7x = (3 + 7)x = 10x$$
$$7y^2 - y^2 = 7y^2 - 1y^2 = (7 - 1)y^2 = 6y^2.$$

This process is called **combining like terms**.

GREAT QUESTION!

What is the bottom line for combining like terms?

To combine like terms mentally, add or subtract the coefficients of the terms. Use this result as the coefficient of the terms' variable factor(s).

An algebraic expression is **simplified** when parentheses have been removed and like terms have been combined.

EXAMPLE 8 Simplifying an Algebraic Expression

Simplify: $6(2x^2 + 4x) + 10(4x^2 + 3x)$.

SOLUTION

$$
\begin{aligned}
&6(2x^2 + 4x) + 10(4x^2 + 3x) \\
&= 6 \cdot 2x^2 + 6 \cdot 4x + 10 \cdot 4x^2 + 10 \cdot 3x && \text{Use the distributive property to remove the parentheses.} \\
&= 12x^2 + 24x + 40x^2 + 30x && \text{Multiply.} \\
&= (12x^2 + 40x^2) + (24x + 30x) && \text{Group like terms.} \\
&= 52x^2 + 54x && \text{Combine like terms.}
\end{aligned}
$$

$52x^2$ and $54x$ are not like terms. They contain different variable factors, x^2 and x, and cannot be combined.

Check Point 8 Simplify: $7(4x^2 + 3x) + 2(5x^2 + x)$.

Properties of Negatives

The distributive property can be extended to cover more than two terms within parentheses. For example,

This sign represents subtraction.

This sign tells us that the number is negative.

$$
\begin{aligned}
-3(4x - 2y + 6) &= -3 \cdot 4x - (-3) \cdot 2y - 3 \cdot 6 \\
&= -12x - (-6y) - 18 \\
&= -12x + 6y - 18.
\end{aligned}
$$

The voice balloons illustrate that negative signs can appear side by side. They can represent the operation of subtraction or the fact that a real number is negative. Here is a list of properties of negatives and how they are applied to algebraic expressions:

Properties of Negatives

Let a and b represent real numbers, variables, or algebraic expressions.

Property	Examples
1. $(-1)a = -a$	$(-1)4xy = -4xy$
2. $-(-a) = a$	$-(-6y) = 6y$
3. $(-a)b = -ab$	$(-7)4xy = -7 \cdot 4xy = -28xy$
4. $a(-b) = -ab$	$5x(-3y) = -5x \cdot 3y = -15xy$
5. $-(a + b) = -a - b$	$-(7x + 6y) = -7x - 6y$
6. $-(a - b) = -a + b$ $= b - a$	$-(3x - 7y) = -3x + 7y$ $= 7y - 3x$

It is not uncommon to see algebraic expressions with parentheses preceded by a negative sign or subtraction. Properties 5 and 6 in the box, $-(a + b) = -a - b$ and $-(a - b) = -a + b$, are related to this situation. An expression of the form $-(a + b)$ can be simplified as follows:

$$-(a + b) = -1(a + b) = (-1)a + (-1)b = -a + (-b) = -a - b.$$

Do you see a fast way to obtain the simplified expression on the right in the preceding equation? **If a negative sign or a subtraction symbol appears outside parentheses, drop the parentheses and change the sign of every term within the parentheses.** For example,

$$-(3x^2 - 7x - 4) = -3x^2 + 7x + 4.$$

EXAMPLE 9 Simplifying an Algebraic Expression

Simplify: $8x + 2[5 - (x - 3)]$.

SOLUTION

$$\begin{aligned} &8x + 2[5 - (x - 3)] && \\ &= 8x + 2[5 - x + 3] && \text{Drop parentheses and change the sign of each term in parentheses: } -(x - 3) = -x + 3. \\ &= 8x + 2[8 - x] && \text{Simplify inside brackets: } 5 + 3 = 8. \\ &= 8x + 16 - 2x && \text{Apply the distributive property: } 2[8 - x] = 2 \cdot 8 - 2x = 16 - 2x. \\ &= (8x - 2x) + 16 && \text{Group like terms.} \\ &= (8 - 2)x + 16 && \text{Apply the distributive property.} \\ &= 6x + 16 && \text{Simplify.} \end{aligned}$$

Check Point 9 Simplify: $6 + 4[7 - (x - 2)]$.

Blitzer Bonus || Using Algebra to Measure Blood-Alcohol Concentration

The amount of alcohol in a person's blood is known as blood-alcohol concentration (BAC), measured in grams of alcohol per deciliter of blood. A BAC of 0.08, meaning 0.08%, indicates that a person has 8 parts alcohol per 10,000 parts blood. In every state in the United States, it is illegal to drive with a BAC of 0.08 or higher.

How Do I Measure My Blood-Alcohol Concentration?

Here's a formula that models BAC for a person who weighs w pounds and who has n drinks* per hour.

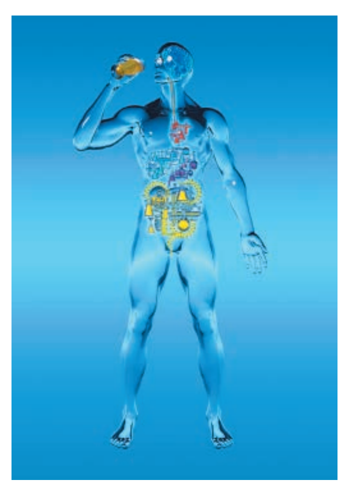
$$\underbrace{\text{BAC}}_{\text{Blood-alcohol concentration}} = \frac{600\overbrace{n}^{\text{Number of drinks consumed in an hour}}}{\underbrace{w}_{\text{Body weight, in pounds}}(0.6n + 169)}$$

*A drink can be a 12-ounce can of beer, a 5-ounce glass of wine, or a 1.5-ounce shot of liquor. Each contains approximately 14 grams, or $\frac{1}{2}$ ounce, of alcohol.

Blood-alcohol concentration can be used to quantify the meaning of "tipsy."

BAC	Effects on Behavior
0.05	Feeling of well-being; mild release of inhibitions; absence of observable effects
0.08	Feeling of relaxation; mild sedation; exaggeration of emotions and behavior; slight impairment of motor skills; increase in reaction time
0.12	Muscle control and speech impaired; difficulty performing motor skills; uncoordinated behavior
0.15	Euphoria; major impairment of physical and mental functions; irresponsible behavior; some difficulty standing, walking, and talking
0.35	Surgical anesthesia; lethal dosage for a small percentage of people
0.40	Lethal dosage for 50% of people; severe circulatory and respiratory depression; alcohol poisoning/overdose

Source: National Clearinghouse for Alcohol and Drug Information



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Keeping in mind the meaning of "tipsy," we can use our model to compare blood-alcohol concentrations of a 120-pound person and a 200-pound person for various numbers of drinks. We determined each BAC using a calculator, rounding to three decimal places.

Blood-Alcohol Concentrations of a 120-Pound Person

$$\text{BAC} = \frac{600n}{120(0.6n + 169)}$$

n (number of drinks per hour)	1	2	3	4	5	6	7	8	9	10
BAC (blood-alcohol concentration)	0.029	0.059	0.088	0.117	0.145	0.174	0.202	0.230	0.258	0.286

Illegal to drive

Blood-Alcohol Concentrations of a 200-Pound Person

$$\text{BAC} = \frac{600n}{200(0.6n + 169)}$$

n (number of drinks per hour)	1	2	3	4	5	6	7	8	9	10
BAC (blood-alcohol concentration)	0.018	0.035	0.053	0.070	0.087	0.104	0.121	0.138	0.155	0.171

Illegal to drive

Like all mathematical models, the formula for BAC gives approximate rather than exact values. There are other variables that influence blood-alcohol concentration that are not contained in the model. These include the rate at which an individual's body processes alcohol, how quickly one drinks, sex, age, physical condition, and the amount of food eaten prior to drinking.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. A combination of numbers, variables, and operation symbols is called an algebraic __________.

2. If n is a counting number, b^n, read ______________, indicates that there are n factors of b. The number b is called the ______ and the number n is called the ________.

3. An equation that expresses a relationship between two or more variables, such as $H = \frac{9}{10}(220 - a)$, is called a/an ________. The process of finding such equations to describe real-world phenomena is called mathematical ________. Such equations, together with the meaning assigned to the variables, are called mathematical ________.

4. The set of elements common to both set A and set B is called the __________ of sets A and B, and is symbolized by ______.

5. The set of elements that are members of set A or set B or of both sets is called the ______ of sets A and B and is symbolized by ______.

6. The set $\{1, 2, 3, 4, 5, \ldots\}$ is called the set of ______ numbers.

7. The set $\{0, 1, 2, 3, 4, 5, \ldots\}$ is called the set of ______ numbers

8. The set $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ is called the set of ______.

9. The set of numbers in the form $\frac{a}{b}$, where a and b belong to the set in Exercise 8 and $b \neq 0$, is called the set of ________ numbers.

10. The set of numbers whose decimal representations are neither terminating nor repeating is called the set of ________ numbers.

11. Every real number is either a/an ________ number or a/an ________ number.

12. The notation $|x|$ is read the __________ of x. If $x \geq 0$, then $|x| =$ ______. If $x < 0$, then $|x| =$ ______.

13. The commutative properties state that $a + b =$ ______ and $ab =$ ______.

14. The associative properties state that $(a + b) + c =$ __________ and ______ $= a(bc)$.

15. The distributive property states that $a(b + c) =$ ______.

16. $a + (-a) =$ ______: The sum of a real number and its additive ______ is ______, the additive ______.

17. $a \cdot \frac{1}{a} = 1, a \neq 0$: The product of a nonzero real number and its multiplicative ______ is ______, the multiplicative ______.

18. An algebraic expression is ________ when parentheses have been removed and like terms have been combined.

19. $-(-a) =$ ______.

EXERCISE SET P.1

Practice Exercises

In Exercises 1–16, evaluate each algebraic expression for the given value or values of the variable(s).

1. $7 + 5x$, for $x = 10$
2. $8 + 6x$, for $x = 5$
3. $6x - y$, for $x = 3$ and $y = 8$
4. $8x - y$, for $x = 3$ and $y = 4$
5. $x^2 + 3x$, for $x = 8$
6. $x^2 + 5x$, for $x = 6$
7. $x^2 - 6x + 3$, for $x = 7$
8. $x^2 - 7x + 4$, for $x = 8$
9. $4 + 5(x - 7)^3$, for $x = 9$
10. $6 + 5(x - 6)^3$, for $x = 8$
11. $x^2 - 3(x - y)$, for $x = 8$ and $y = 2$
12. $x^2 - 4(x - y)$, for $x = 8$ and $y = 3$
13. $\dfrac{5(x + 2)}{2x - 14}$, for $x = 10$
14. $\dfrac{7(x - 3)}{2x - 16}$, for $x = 9$
15. $\dfrac{2x + 3y}{x + 1}$, for $x = -2$ and $y = 4$
16. $\dfrac{2x + y}{xy - 2x}$, for $x = -2$ and $y = 4$

The formula

$$C = \frac{5}{9}(F - 32)$$

expresses the relationship between Fahrenheit temperature, F, and Celsius temperature, C. In Exercises 17–18, use the formula to convert the given Fahrenheit temperature to its equivalent temperature on the Celsius scale.

17. 50°F **18.** 86°F

A football was kicked vertically upward from a height of 4 feet with an initial speed of 60 feet per second. The formula

$$h = 4 + 60t - 16t^2$$

describes the ball's height above the ground, h, in feet, t seconds after it was kicked. Use this formula to solve Exercises 19–20.

19. What was the ball's height 2 seconds after it was kicked?
20. What was the ball's height 3 seconds after it was kicked?

In Exercises 21–28, find the intersection of the sets.

21. $\{1, 2, 3, 4\} \cap \{2, 4, 5\}$ **22.** $\{1, 3, 7\} \cap \{2, 3, 8\}$
23. $\{s, e, t\} \cap \{t, e, s\}$ **24.** $\{r, e, a, l\} \cap \{l, e, a, r\}$
25. $\{1, 3, 5, 7\} \cap \{2, 4, 6, 8, 10\}$
26. $\{0, 1, 3, 5\} \cap \{-5, -3, -1\}$
27. $\{a, b, c, d\} \cap \varnothing$ **28.** $\{w, y, z\} \cap \varnothing$

In Exercises 29–34, find the union of the sets.

29. $\{1, 2, 3, 4\} \cup \{2, 4, 5\}$ **30.** $\{1, 3, 7, 8\} \cup \{2, 3, 8\}$
31. $\{1, 3, 5, 7\} \cup \{2, 4, 6, 8, 10\}$ **32.** $\{0, 1, 3, 5\} \cup \{2, 4, 6\}$
33. $\{a, e, i, o, u\} \cup \varnothing$ **34.** $\{e, m, p, t, y\} \cup \varnothing$

In Exercises 35–38, list all numbers from the given set that are **a.** *natural numbers,* **b.** *whole numbers,* **c.** *integers,* **d.** *rational numbers,* **e.** *irrational numbers,* **f.** *real numbers.*

35. $\{-9, -\frac{4}{5}, 0, 0.25, \sqrt{3}, 9.2, \sqrt{100}\}$
36. $\{-7, -0.\overline{6}, 0, \sqrt{49}, \sqrt{50}\}$
37. $\{-11, -\frac{5}{6}, 0, 0.75, \sqrt{5}, \pi, \sqrt{64}\}$
38. $\{-5, -0.\overline{3}, 0, \sqrt{2}, \sqrt{4}\}$
39. Give an example of a whole number that is not a natural number.
40. Give an example of a rational number that is not an integer.
41. Give an example of a number that is an integer, a whole number, and a natural number.
42. Give an example of a number that is a rational number, an integer, and a real number.

Determine whether each statement in Exercises 43–50 is true or false.

43. $-13 \le -2$ **44.** $-6 > 2$
45. $4 \ge -7$ **46.** $-13 < -5$
47. $-\pi \ge -\pi$ **48.** $-3 > -13$
49. $0 \ge -6$ **50.** $0 \ge -13$

In Exercises 51–60, rewrite each expression without absolute value bars.

51. $|300|$ **52.** $|-203|$
53. $|12 - \pi|$ **54.** $|7 - \pi|$
55. $|\sqrt{2} - 5|$ **56.** $|\sqrt{5} - 13|$
57. $\dfrac{-3}{|-3|}$ **58.** $\dfrac{-7}{|-7|}$
59. $||-3| - |-7||$ **60.** $||-5| - |-13||$

In Exercises 61–66, evaluate each algebraic expression for $x = 2$ and $y = -5$.

61. $|x + y|$ **62.** $|x - y|$
63. $|x| + |y|$ **64.** $|x| - |y|$
65. $\dfrac{y}{|y|}$ **66.** $\dfrac{|x|}{x} + \dfrac{|y|}{y}$

In Exercises 67–74, express the distance between the given numbers using absolute value. Then find the distance by evaluating the absolute value expression.

67. 2 and 17 **68.** 4 and 15
69. -2 and 5 **70.** -6 and 8
71. -19 and -4 **72.** -26 and -3
73. -3.6 and -1.4 **74.** -5.4 and -1.2

In Exercises 75–84, state the name of the property illustrated.

75. $6 + (-4) = (-4) + 6$
76. $11 \cdot (7 + 4) = 11 \cdot 7 + 11 \cdot 4$
77. $6 + (2 + 7) = (6 + 2) + 7$
78. $6 \cdot (2 \cdot 3) = 6 \cdot (3 \cdot 2)$
79. $(2 + 3) + (4 + 5) = (4 + 5) + (2 + 3)$
80. $7 \cdot (11 \cdot 8) = (11 \cdot 8) \cdot 7$
81. $2(-8 + 6) = -16 + 12$
82. $-8(3 + 11) = -24 + (-88)$

83. $\frac{1}{(x+3)}(x+3) = 1, x \neq -3$

84. $(x+4) + [-(x+4)] = 0$

In Exercises 85–96, simplify each algebraic expression.

85. $5(3x+4) - 4$

86. $2(5x+4) - 3$

87. $5(3x-2) + 12x$

88. $2(5x-1) + 14x$

89. $7(3y-5) + 2(4y+3)$

90. $4(2y-6) + 3(5y+10)$

91. $5(3y-2) - (7y+2)$

92. $4(5y-3) - (6y+3)$

93. $7 - 4[3 - (4y-5)]$

94. $6 - 5[8 - (2y-4)]$

95. $18x^2 + 4 - [6(x^2-2) + 5]$

96. $14x^2 + 5 - [7(x^2-2) + 4]$

In Exercises 97–102, write each algebraic expression without parentheses.

97. $-(-14x)$

98. $-(-17y)$

99. $-(2x - 3y - 6)$

100. $-(5x - 13y - 1)$

101. $\frac{1}{3}(3x) + [(4y) + (-4y)]$

102. $\frac{1}{2}(2y) + [(-7x) + 7x]$

Practice Plus

In Exercises 103–110, insert either $<$, $>$, *or* $=$ *in the shaded area to make a true statement.*

103. $|-6| \;\square\; |-3|$

104. $|-20| \;\square\; |-50|$

105. $\left|\frac{3}{5}\right| \;\square\; |-0.6|$

106. $\left|\frac{5}{2}\right| \;\square\; |-2.5|$

107. $\frac{30}{40} - \frac{3}{4} \;\square\; \frac{14}{15} \cdot \frac{15}{14}$

108. $\frac{17}{18} \cdot \frac{18}{17} \;\square\; \frac{50}{60} - \frac{5}{6}$

109. $\frac{8}{13} \div \frac{8}{13} \;\square\; |-1|$

110. $|-2| \;\square\; \frac{4}{17} \div \frac{4}{17}$

In Exercises 111–120, use the order of operations to simplify each expression.

111. $8^2 - 16 \div 2^2 \cdot 4 - 3$

112. $10^2 - 100 \div 5^2 \cdot 2 - 3$

113. $\frac{5 \cdot 2 - 3^2}{[3^2 - (-2)]^2}$

114. $\frac{10 \div 2 + 3 \cdot 4}{(12 - 3 \cdot 2)^2}$

115. $8 - 3[-2(2-5) - 4(8-6)]$

116. $8 - 3[-2(5-7) - 5(4-2)]$

117. $\frac{2(-2) - 4(-3)}{5 - 8}$

118. $\frac{6(-4) - 5(-3)}{9 - 10}$

119. $\frac{(5-6)^2 - 2|3-7|}{89 - 3 \cdot 5^2}$

120. $\frac{12 \div 3 \cdot 5|2^2 + 3^2|}{7 + 3 - 6^2}$

In Exercises 121–128, write each English phrase as an algebraic expression. Then simplify the expression. Let x represent the number.

121. A number decreased by the sum of the number and four

122. A number decreased by the difference between eight and the number

123. Six times the product of negative five and a number

124. Ten times the product of negative four and a number

125. The difference between the product of five and a number and twice the number

126. The difference between the product of six and a number and negative two times the number

127. The difference between eight times a number and six more than three times the number

128. Eight decreased by three times the sum of a number and six

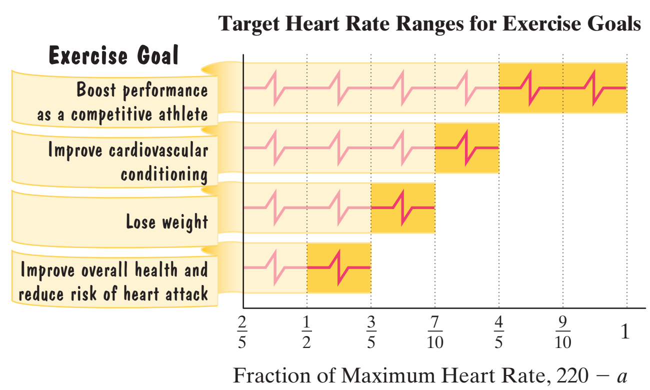
Application Exercises

The maximum heart rate, in beats per minute, that you should achieve during exercise is 220 minus your age:

$$220 - a.$$

This algebraic expression gives maximum heart rate in terms of age, a.

The following bar graph shows the target heart rate ranges for four types of exercise goals. The lower and upper limits of these ranges are fractions of the maximum heart rate, 220 − a. Exercises 129–130 are based on the information in the graph.



129. If your exercise goal is to improve cardiovascular conditioning, the graph shows the following range for target heart rate, H, in beats per minute:

Lower limit of range $\quad H = \frac{7}{10}(220 - a)$

Upper limit of range $\quad H = \frac{4}{5}(220 - a)$.

a. What is the lower limit of the heart rate range, in beats per minute, for a 20-year-old with this exercise goal?

b. What is the upper limit of the heart rate range, in beats per minute, for a 20-year-old with this exercise goal?

130. If your exercise goal is to improve overall health, the graph shows the following range for target heart rate, H, in beats per minute:

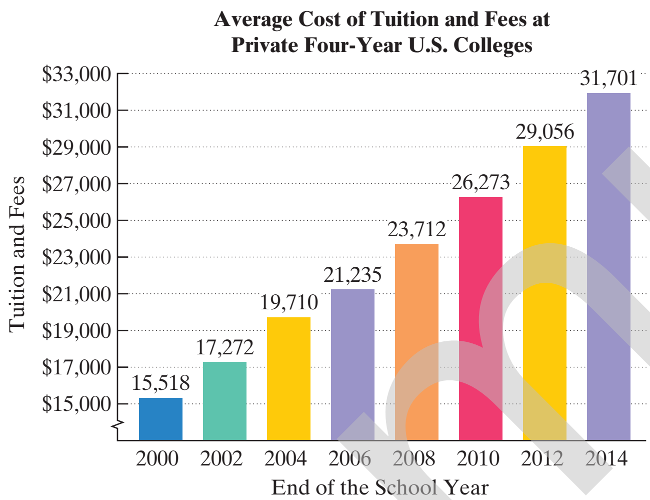
Lower limit of range $\quad H = \frac{1}{2}(220 - a)$

Upper limit of range $\quad H = \frac{3}{5}(220 - a)$.

a. What is the lower limit of the heart rate range, in beats per minute, for a 30-year-old with this exercise goal?

b. What is the upper limit of the heart rate range, in beats per minute, for a 30-year-old with this exercise goal?

The bar graph shows the average cost of tuition and fees at private four-year colleges in the United States.



Source: The College Board

The formula

$$T = 21x^2 + 862x + 15{,}552$$

models the average cost of tuition and fees, T, at private U.S. colleges for the school year ending x years after 2000. Use this information to solve Exercises 131–132.

131. **a.** Use the formula to find the average cost of tuition and fees at private U.S. colleges for the school year ending in 2014.

b. By how much does the formula underestimate or overestimate the actual cost shown by the graph for the school year ending in 2014?

c. Use the formula to project the average cost of tuition and fees at private U.S. colleges for the school year ending in 2020.

132. **a.** Use the formula to find the average cost of tuition and fees at private U.S. colleges for the school year ending in 2012.

b. By how much does the formula underestimate or overestimate the actual cost shown by the graph for the school year ending in 2012?

c. Use the formula to project the average cost of tuition and fees at private U.S. colleges for the school year ending in 2022.

133. You had $10,000 to invest. You put x dollars in a safe, government-insured certificate of deposit paying 5% per year. You invested the remainder of the money in noninsured corporate bonds paying 12% per year. Your total interest earned at the end of the year is given by the algebraic expression

$$0.05x + 0.12(10{,}000 - x).$$

a. Simplify the algebraic expression.

b. Use each form of the algebraic expression to determine your total interest earned at the end of the year if you invested $6000 in the safe, government-insured certificate of deposit.

134. It takes you 50 minutes to get to campus. You spend t minutes walking to the bus stop and the rest of the time riding the bus. Your walking rate is 0.06 mile per minute and the bus travels at a rate of 0.5 mile per minute. The total distance walking and traveling by bus is given by the algebraic expression

$$0.06t + 0.5(50 - t).$$

a. Simplify the algebraic expression.

b. Use each form of the algebraic expression to determine the total distance that you travel if you spend 20 minutes walking to the bus stop.

135. Read the Blitzer Bonus beginning on page 15. Use the formula

$$\text{BAC} = \frac{600n}{w(0.6n + 169)}$$

and replace w with your body weight. Using this formula and a calculator, compute your BAC for integers from $n = 1$ to $n = 10$. Round to three decimal places. According to this model, how many drinks can you consume in an hour without exceeding the legal measure of drunk driving?

Explaining the Concepts

ACHIEVING SUCCESS

An effective way to understand something is to explain it to someone else. You can do this by using the Explaining the Concepts exercises that ask you to respond with verbal or written explanations. Speaking or writing about a new concept uses a different part of your brain than thinking about the concept. Explaining new ideas verbally will quickly reveal any gaps in your understanding. It will also help you to remember new concepts for longer periods of time.

136. What is an algebraic expression? Give an example with your explanation.

137. If n is a natural number, what does b^n mean? Give an example with your explanation.

138. What does it mean when we say that a formula models real-world phenomena?

139. What is the intersection of sets A and B?

140. What is the union of sets A and B?

141. How do the whole numbers differ from the natural numbers?

142. Can a real number be both rational and irrational? Explain your answer.

143. If you are given two real numbers, explain how to determine which is the lesser.

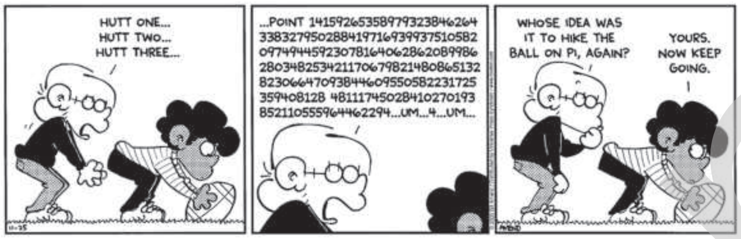
Critical Thinking Exercises

Make Sense? *In Exercises 144–147, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

144. My mathematical model describes the data for tuition and fees at public four-year colleges for the past ten years extremely well, so it will serve as an accurate prediction for the cost of public colleges in 2050.

145. A model that describes the average cost of tuition and fees at private U.S. colleges for the school year ending x years after 2000 cannot be used to estimate the cost of private education for the school year ending in 2000.

146. The humor in this cartoon is based on the fact that the football will never be hiked.



Foxtrot © 2003, 2009 by Bill Amend/Used by permission of Universal Uclick.

147. Just as the commutative properties change groupings, the associative properties change order.

In Exercises 148–155, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

148. Every rational number is an integer.

149. Some whole numbers are not integers.

150. Some rational numbers are not positive.

151. Irrational numbers cannot be negative.

152. The term x has no coefficient.

153. $5 + 3(x - 4) = 8(x - 4) = 8x - 32$

154. $-x - x = -x + (-x) = 0$

155. $x - 0.02(x + 200) = 0.98x - 4$

In Exercises 156–158, insert either < or > in the shaded area between the numbers to make the statement true.

156. $\sqrt{2}$ ▢ 1.5

157. $-\pi$ ▢ -3.5

158. $-\dfrac{3.14}{2}$ ▢ $-\dfrac{\pi}{2}$

Preview Exercises

Exercises 159–161 will help you prepare for the material covered in the next section.

159. In parts (a) and (b), complete each statement.

a. $b^4 \cdot b^3 = (b \cdot b \cdot b \cdot b)(b \cdot b \cdot b) = b^?$

b. $b^5 \cdot b^5 = (b \cdot b \cdot b \cdot b \cdot b)(b \cdot b \cdot b \cdot b \cdot b) = b^?$

c. Generalizing from parts (a) and (b), what should be done with the exponents when multiplying exponential expressions with the same base?

160. In parts (a) and (b), complete each statement.

a. $\dfrac{b^7}{b^3} = \dfrac{\cancel{b} \cdot \cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b \cdot b}{\cancel{b} \cdot \cancel{b} \cdot \cancel{b}} = b^?$

b. $\dfrac{b^8}{b^2} = \dfrac{\cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b}{\cancel{b} \cdot \cancel{b}} = b^?$

c. Generalizing from parts (a) and (b), what should be done with the exponents when dividing exponential expressions with the same base?

161. If 6.2 is multiplied by 10^3, what does this multiplication do to the decimal point in 6.2?

Section P.2 Exponents and Scientific Notation

What am I supposed to learn?

After studying this section, you should be able to:

1. Use the product rule.
2. Use the quotient rule.
3. Use the zero-exponent rule.
4. Use the negative-exponent rule.
5. Use the power rule.
6. Find the power of a product.
7. Find the power of a quotient.
8. Simplify exponential expressions.
9. Use scientific notation.

Bigger than the biggest thing ever and then some. Much bigger than that in fact, really amazingly immense, a totally stunning size, real 'wow, that's big', time … Gigantic multiplied by colossal multiplied by staggeringly huge is the sort of concept we're trying to get across here.

Douglas Adams, *The Restaurant at the End of the Universe*

Although Adams's description may not quite apply to this \$18.9 trillion national debt, exponents can be used to explore the meaning of this "staggeringly huge" number. In this section, you will learn to use exponents to provide a way of putting large and small numbers in perspective.

The Product and Quotient Rules

We have seen that exponents are used to indicate repeated multiplication. Now consider the multiplication of two exponential expressions, such as $b^4 \cdot b^3$. We are multiplying 4 factors of b and 3 factors of b. We have a total of 7 factors of b:

$$b^4 \cdot b^3 = \underbrace{\overbrace{(b \cdot b \cdot b \cdot b)}^{\text{4 factors of } b}\overbrace{(b \cdot b \cdot b)}^{\text{3 factors of } b}}_{\text{Total: 7 factors of } b} = b^7.$$

The product is exactly the same if we add the exponents:

$$b^4 \cdot b^3 = b^{4+3} = b^7.$$

The fact that $b^4 \cdot b^3 = b^7$ suggests the following rule:

1 Use the product rule.

The Product Rule

$$b^m \cdot b^n = b^{m+n}$$

When multiplying exponential expressions with the same base, add the exponents. Use this sum as the exponent of the common base.

EXAMPLE 1 Using the Product Rule

Multiply each expression using the product rule:

a. $2^2 \cdot 2^3$ **b.** $(6x^4y^3)(5x^2y^7)$.

SOLUTION

a. $2^2 \cdot 2^3 = 2^{2+3} = 2^5$ or 32 $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

b. $(6x^4y^3)(5x^2y^7)$

$= 6 \cdot 5 \cdot x^4 \cdot x^2 \cdot y^3 \cdot y^7$ Use the associative and commutative properties. This step can be done mentally.

$= 30x^{4+2}y^{3+7}$

$= 30x^6y^{10}$

Check Point 1 Multiply each expression using the product rule:

a. $3^3 \cdot 3^2$ **b.** $(4x^3y^4)(10x^2y^6)$.

---

2 Use the quotient rule.

Now, consider the division of two exponential expressions, such as the quotient of b^7 and b^3. We are dividing 7 factors of b by 3 factors of b.

$$\frac{b^7}{b^3} = \frac{b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b}{b \cdot b \cdot b} = \underbrace{\boxed{\frac{b \cdot b \cdot b}{b \cdot b \cdot b}}}_{\text{This factor is equal to 1.}} \cdot b \cdot b \cdot b \cdot b = 1 \cdot b \cdot b \cdot b \cdot b = b^4$$

The quotient is exactly the same if we subtract the exponents:

$$\frac{b^7}{b^3} = b^{7-3} = b^4.$$

This suggests the following rule:

The Quotient Rule

$$\frac{b^m}{b^n} = b^{m-n}, \quad b \neq 0$$

When dividing exponential expressions with the same nonzero base, subtract the exponent in the denominator from the exponent in the numerator. Use this difference as the exponent of the common base.

EXAMPLE 2 Using the Quotient Rule

Divide each expression using the quotient rule:

a. $\frac{(-2)^7}{(-2)^4}$ **b.** $\frac{30x^{12}y^9}{5x^3y^7}$.

SOLUTION

a. $\frac{(-2)^7}{(-2)^4} = (-2)^{7-4} = (-2)^3$ or -8

$(-2)^3 = (-2)(-2)(-2) = -8$

b. $\frac{30x^{12}y^9}{5x^3y^7} = \frac{30}{5} \cdot \frac{x^{12}}{x^3} \cdot \frac{y^9}{y^7} = 6x^{12-3}y^{9-7} = 6x^9y^2$

Check Point 2 Divide each expression using the quotient rule:

a. $\frac{(-3)^6}{(-3)^3}$ **b.** $\frac{27x^{14}y^8}{3x^3y^5}$.

3 Use the zero-exponent rule.

Zero as an Exponent

A nonzero base can be raised to the 0 power. The quotient rule can be used to help determine what zero as an exponent should mean. Consider the quotient of b^4 and b^4, where b is not zero. We can determine this quotient in two ways.

$$\frac{b^4}{b^4} = 1 \qquad \frac{b^4}{b^4} = b^{4-4} = b^0$$

Any nonzero expression divided by itself is 1.

Use the quotient rule and subtract exponents.

This means that b^0 must equal 1.

The Zero-Exponent Rule

If b is any real number other than 0,

$$b^0 = 1.$$

Here are examples involving simplification using the zero-exponent rule:

$$8^0 = 1, \quad (-6)^0 = 1, \quad -6^0 = -1, \quad (5x)^0 = 1, \quad 5x^0 = 5.$$

Because there are no parentheses, only 6 is raised to the 0 power: $-6^0 = -(6^0) = -1$.

Because there are no parentheses, only x is raised to the 0 power: $5x^0 = 5 \cdot 1 = 5$.

4 Use the negative-exponent rule.

Negative Integers as Exponents

A nonzero base can be raised to a negative power. The quotient rule can be used to help determine what a negative integer as an exponent should mean. Consider the quotient of b^3 and b^5, where b is not zero. We can determine this quotient in two ways.

$$\frac{b^3}{b^5} = \frac{\cancel{b}\cdot\cancel{b}\cdot\cancel{b}}{\cancel{b}\cdot\cancel{b}\cdot\cancel{b}\cdot b\cdot b} = \frac{1}{b^2} \qquad \frac{b^3}{b^5} = b^{3-5} = b^{-2}$$

After dividing common factors, we have two factors of b in the denominator.

Use the quotient rule and subtract exponents.

Notice that $\frac{b^3}{b^5}$ equals both b^{-2} and $\frac{1}{b^2}$. This means that b^{-2} must equal $\frac{1}{b^2}$. This example is a special case of the **negative-exponent rule**.

The Negative-Exponent Rule

If b is any real number other than 0 and n is a natural number, then

$$b^{-n} = \frac{1}{b^n}.$$

EXAMPLE 3 Using the Negative-Exponent Rule

Use the negative-exponent rule to write each expression with a positive exponent. Simplify, if possible:

a. 9^{-2} **b.** $(-2)^{-5}$ **c.** $\frac{1}{6^{-2}}$ **d.** $7x^{-5}y^2$.

SOLUTION

a. $9^{-2} = \frac{1}{9^2} = \frac{1}{81}$

b. $(-2)^{-5} = \frac{1}{(-2)^5} = \frac{1}{(-2)(-2)(-2)(-2)(-2)} = \frac{1}{-32} = -\frac{1}{32}$

Only the sign of the exponent, −5, changes. The base, −2, does not change sign.

c. $\frac{1}{6^{-2}} = \frac{1}{\frac{1}{6^2}} = 1\cdot\frac{6^2}{1} = 6^2 = 36$

d. $7x^{-5}y^2 = 7\cdot\frac{1}{x^5}\cdot y^2 = \frac{7y^2}{x^5}$

Check Point 3 Use the negative-exponent rule to write each expression with a positive exponent. Simplify, if possible:

a. 5^{-2} **b.** $(-3)^{-3}$ **c.** $\frac{1}{4^{-2}}$ **d.** $3x^{-6}y^4$.

In Example 3 and Check Point 3, did you notice that

$$\frac{1}{6^{-2}} = 6^2 \quad \text{and} \quad \frac{1}{4^{-2}} = 4^2?$$

In general, if a negative exponent appears in a denominator, an expression can be written with a positive exponent using

$$\frac{1}{b^{-n}} = b^n.$$

Negative Exponents in Numerators and Denominators

If b is any real number other than 0 and n is a natural number, then

$$b^{-n} = \frac{1}{b^n} \quad \text{and} \quad \frac{1}{b^{-n}} = b^n.$$

When a negative number appears as an exponent, switch the position of the base (from numerator to denominator or from denominator to numerator) and make the exponent positive. The sign of the base does not change.

5 Use the power rule.

The Power Rule for Exponents (Powers to Powers)

The next property of exponents applies when an exponential expression is raised to a power. Here is an example:

$$\underbrace{(b^2)^4}_{\text{The exponential expression } b^2 \text{ is raised to the fourth power.}}.$$

There are four factors of b^2. Thus,

$$(b^2)^4 = b^2 \cdot b^2 \cdot b^2 \cdot b^2 = \underbrace{b^{2+2+2+2}}_{\text{Add exponents when multiplying with the same base.}} = b^8.$$

We can obtain the answer, b^8, by multiplying the original exponents:

$$(b^2)^4 = b^{2\cdot4} = b^8.$$

This suggests the following rule:

The Power Rule (Powers to Powers)

$$(b^m)^n = b^{mn}$$

When an exponential expression is raised to a power, multiply the exponents. Place the product of the exponents on the base and remove the parentheses.

EXAMPLE 4 Using the Power Rule (Powers to Powers)

Simplify each expression using the power rule:

a. $(2^2)^3$ **b.** $(y^5)^{-3}$ **c.** $(b^{-4})^{-2}$.

SOLUTION

a. $(2^2)^3 = 2^{2\cdot3} = 2^6$ or 64 **b.** $(y^5)^{-3} = y^{5(-3)} = y^{-15} = \frac{1}{y^{15}}$

c. $(b^{-4})^{-2} = b^{(-4)(-2)} = b^8$

Check Point 4 Simplify each expression using the power rule:

a. $(3^3)^2$ **b.** $(y^7)^{-2}$ **c.** $(b^{-3})^{-4}$.

6 Find the power of a product.

The Products-to-Powers Rule for Exponents

The next property of exponents applies when we are raising a product to a power. Here is an example:

$$(2x)^4.$$

The product $2x$ is raised to the fourth power.

There are four factors of $2x$. Thus,

$$(2x)^4 = 2x \cdot 2x \cdot 2x \cdot 2x = 2 \cdot 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x = 2^4 x^4.$$

We can obtain the answer, 2^4x^4, by raising each factor within the parentheses to the fourth power:

$$(2x)^4 = 2^4x^4.$$

The fact that $(2x)^4 = 2^4x^4$ suggests the following rule:

Products to Powers

$$(ab)^n = a^n b^n$$

When a product is raised to a power, raise each factor to that power.

EXAMPLE 5 Raising a Product to a Power

Simplify: $(-2y^2)^4$.

SOLUTION

$$(-2y^2)^4 = (-2)^4(y^2)^4$$ Raise each factor to the fourth power.

$$= (-2)^4 y^{2\cdot 4}$$ To raise an exponential expression to a power, multiply exponents: $(b^m)^n = b^{mn}$.

$$= 16y^8$$ Simplify: $(-2)^4 = (-2)(-2)(-2)(-2) = 16$.

Check Point 5 Simplify: $(-4x)^3$.

The rule for raising a product to a power can be extended to cover three or more factors. For example,

$$(-2xy)^3 = (-2)^3x^3y^3 = -8x^3y^3.$$

7 Find the power of a quotient.

The Quotients-to-Powers Rule for Exponents

The following rule is used to raise a quotient to a power:

Quotients to Powers

If b is a nonzero real number, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

When a quotient is raised to a power, raise the numerator to that power and divide by the denominator to that power.

EXAMPLE 6 Raising Quotients to Powers

Simplify by raising each quotient to the given power:

a. $\left(-\frac{3}{x}\right)^4$ **b.** $\left(\frac{x^2}{4}\right)^3$.

SOLUTION

a. $\left(-\frac{3}{x}\right)^4 = \frac{(-3)^4}{x^4} = \frac{(-3)(-3)(-3)(-3)}{x^4} = \frac{81}{x^4}$

b. $\left(\frac{x^2}{4}\right)^3 = \frac{(x^2)^3}{4^3} = \frac{x^{2\cdot 3}}{4\cdot 4\cdot 4} = \frac{x^6}{64}$

Check Point 6 Simplify:

a. $\left(-\frac{2}{y}\right)^5$ **b.** $\left(\frac{x^5}{3}\right)^3$.

8 Simplify exponential expressions.

Simplifying Exponential Expressions

Properties of exponents are used to simplify exponential expressions. An exponential expression is **simplified** when

- No parentheses appear.
- No powers are raised to powers.
- Each base occurs only once.
- No negative or zero exponents appear.

Simplifying Exponential Expressions

	Example
1. If necessary, remove parentheses by using $(ab)^n = a^n b^n$ or $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.	$(xy)^3 = x^3y^3$
2. If necessary, simplify powers to powers by using $(b^m)^n = b^{mn}$.	$(x^4)^3 = x^{4\cdot 3} = x^{12}$
3. If necessary, be sure that each base appears only once by using $b^m \cdot b^n = b^{m+n}$ or $\frac{b^m}{b^n} = b^{m-n}$.	$x^4 \cdot x^3 = x^{4+3} = x^7$
4. If necessary, rewrite exponential expressions with zero powers as 1 ($b^0 = 1$). Furthermore, write the answer with positive exponents by using $b^{-n} = \frac{1}{b^n}$ or $\frac{1}{b^{-n}} = b^n$.	$\frac{x^5}{x^8} = x^{5-8} = x^{-3} = \frac{1}{x^3}$

The following example shows how to simplify exponential expressions. Throughout the example, assume that no variable in a denominator is equal to zero.

EXAMPLE 7 Simplifying Exponential Expressions

Simplify:

a. $(-3x^4y^5)^3$ **b.** $(-7xy^4)(-2x^5y^6)$ **c.** $\frac{-35x^2y^4}{5x^6y^{-8}}$ **d.** $\left(\frac{4x^2}{y}\right)^{-3}$.

SOLUTION

a. $(-3x^4y^5)^3 = (-3)^3(x^4)^3(y^5)^3$ Raise each factor inside the parentheses to the third power.

$= (-3)^3x^{4\cdot 3}y^{5\cdot 3}$ Multiply the exponents when raising powers to powers.

$= -27x^{12}y^{15}$ $(-3)^3 = (-3)(-3)(-3) = -27$

b. $(-7xy^4)(-2x^5y^6) = (-7)(-2)xx^5y^4y^6$ Group factors with the same base.

$= 14x^{1+5}y^{4+6}$ When multiplying expressions with the same base, add the exponents.

$= 14x^6y^{10}$ Simplify.

c. $\dfrac{-35x^2y^4}{5x^6y^{-8}} = \left(\dfrac{-35}{5}\right)\left(\dfrac{x^2}{x^6}\right)\left(\dfrac{y^4}{y^{-8}}\right)$ Group factors with the same base.

$= -7x^{2-6}y^{4-(-8)}$ When dividing expressions with the same base, subtract the exponents.

$= -7x^{-4}y^{12}$ Simplify. Notice that $4 - (-8) = 4 + 8 = 12$.

$= \dfrac{-7y^{12}}{x^4}$ Write as a fraction and move the base with the negative exponent, x^{-4}, to the other side of the fraction bar and make the negative exponent positive.

d. $\left(\dfrac{4x^2}{y}\right)^{-3} = \dfrac{(4x^2)^{-3}}{y^{-3}}$ Raise the numerator and the denominator to the -3 power.

$= \dfrac{4^{-3}(x^2)^{-3}}{y^{-3}}$ Raise each factor inside the parentheses to the -3 power.

$= \dfrac{4^{-3}x^{-6}}{y^{-3}}$ Multiply the exponents when raising a power to a power: $(x^2)^{-3} = x^{2(-3)} = x^{-6}$.

$= \dfrac{y^3}{4^3x^6}$ Move each base with a negative exponent to the other side of the fraction bar and make each negative exponent positive.

$= \dfrac{y^3}{64x^6}$ $4^3 = 4\cdot 4\cdot 4 = 64$

Check Point 7 Simplify:

a. $(2x^3y^6)^4$ **b.** $(-6x^2y^5)(3xy^3)$

c. $\dfrac{100x^{12}y^2}{20x^{16}y^{-4}}$ **d.** $\left(\dfrac{5x}{y^4}\right)^{-2}$.

GREAT QUESTION!

Simplifying exponential expressions seems to involve lots of steps. Are there common errors I can avoid along the way?

Yes. Here's a list. The first column has the correct simplification. The second column contains common errors you should try to avoid.

Correct	Incorrect	Description of Error
$b^3 \cdot b^4 = b^7$	$b^3 \cdot b^4 = b^{12}$	The exponents should be added, not multiplied.
$3^2 \cdot 3^4 = 3^6$	$3^2 \cdot 3^4 = 9^6$	The common base should be retained, not multiplied.
$\frac{5^{16}}{5^4} = 5^{12}$	$\frac{5^{16}}{5^4} = 5^4$	The exponents should be subtracted, not divided.
$(4a)^3 = 64a^3$	$(4a)^3 = 4a^3$	Both factors should be cubed.
$b^{-n} = \frac{1}{b^n}$	$b^{-n} = -\frac{1}{b^n}$	Only the exponent should change sign.
$(a+b)^{-1} = \frac{1}{a+b}$	$(a+b)^{-1} = \frac{1}{a} + \frac{1}{b}$	The exponent applies to the entire expression $a + b$.

9 Use scientific notation.

Scientific Notation

Earth is a 4.5-billion-year-old ball of rock orbiting the Sun. Because a billion is 10^9 (see **Table P.3**), the age of our world can be expressed as

$$4.5 \times 10^9.$$

The number 4.5×10^9 is written in a form called *scientific notation*.

Table P.3 Names of Large Numbers

10^2	hundred
10^3	thousand
10^6	million
10^9	billion
10^{12}	trillion
10^{15}	quadrillion
10^{18}	quintillion
10^{21}	sextillion
10^{24}	septillion
10^{27}	octillion
10^{30}	nonillion
10^{100}	googol
10^{googol}	googolplex

Scientific Notation

A number is written in **scientific notation** when it is expressed in the form

$$a \times 10^n,$$

where the absolute value of a is greater than or equal to 1 and less than 10 ($1 \le |a| < 10$), and n is an integer.

It is customary to use the multiplication symbol, $\times$, rather than a dot, when writing a number in scientific notation.

Converting from Scientific to Decimal Notation

Here are two examples of numbers in scientific notation:

$$6.4 \times 10^5 \quad \text{means} \quad 640{,}000.$$
$$2.17 \times 10^{-3} \quad \text{means} \quad 0.00217.$$

Do you see that the number with the positive exponent is relatively large and the number with the negative exponent is relatively small?

We can use n, the exponent on the 10 in $a \times 10^n$, to change a number in scientific notation to decimal notation. If n is **positive**, move the decimal point in a to the **right** n places. If n is **negative**, move the decimal point in a to the **left** $|n|$ places.

EXAMPLE 8 Converting from Scientific to Decimal Notation

Write each number in decimal notation:

a. 6.2×10^7 **b.** -6.2×10^7 **c.** 2.019×10^{-3} **d.** -2.019×10^{-3}.

SOLUTION

In each case, we use the exponent on the 10 to determine how far to move the decimal point and in which direction. In parts (a) and (b), the exponent is positive, so we move the decimal point to the right. In parts (c) and (d), the exponent is negative, so we move the decimal point to the left.

a. $6.2 \times 10^7 = 62{,}000{,}000$

$n = 7$

Move the decimal point 7 places to the right.

b. $-6.2 \times 10^7 = -62{,}000{,}000$

$n = 7$

Move the decimal point 7 places to the right.

c. $2.019 \times 10^{-3} = 0.002019$

$n = -3$

Move the decimal point $|-3|$ places, or 3 places, to the left.

d. $-2.019 \times 10^{-3} = -0.002019$

$n = -3$

Move the decimal point $|-3|$ places, or 3 places, to the left.

Check Point 8 Write each number in decimal notation:

a. -2.6×10^9 **b.** 3.017×10^{-6}.

Converting from Decimal to Scientific Notation

To convert from decimal notation to scientific notation, we reverse the procedure of Example 8.

Converting from Decimal to Scientific Notation

Write the number in the form $a \times 10^n$.

- Determine a, the numerical factor. Move the decimal point in the given number to obtain a number whose absolute value is between 1 and 10, including 1.
- Determine n, the exponent on 10^n. The absolute value of n is the number of places the decimal point was moved. The exponent n is positive if the decimal point was moved to the left, negative if the decimal point was moved to the right, and 0 if the decimal point was not moved.

EXAMPLE 9 Converting from Decimal Notation to Scientific Notation

Write each number in scientific notation:

a. 34,970,000,000,000 **b.** −34,970,000,000,000

c. 0.0000000000802 **d.** −0.0000000000802.

SOLUTION

a. $34{,}970{,}000{,}000{,}000 = 3.497 \times 10^{13}$

Move the decimal point to get a number whose absolute value is between 1 and 10.

The decimal point was moved 13 places to the left, so $n = 13$.

b. $-34{,}970{,}000{,}000{,}000 = -3.497 \times 10^{13}$

TECHNOLOGY

You can use your calculator's $\boxed{\text{EE}}$ (enter exponent) or $\boxed{\text{EXP}}$ key to convert from decimal to scientific notation. Here is how it's done for 0.0000000000802.

Many Scientific Calculators

Keystrokes

.0000000000802 $\boxed{\text{EE}}$ $\boxed{=}$

Display

8.02 − 11

Many Graphing Calculators

Use the mode setting for scientific notation.

Keystrokes

.0000000000802 $\boxed{\text{ENTER}}$

Display

8.02E − 11

c. $0.\underbrace{0000000000802}_{} = 8.02 \times 10^{-11}$

Move the decimal point to get a number whose absolute value is between 1 and 10.

The decimal point was moved 11 places to the right, so $n = -11$.

d. $-0.\underbrace{0000000000802}_{} = -8.02 \times 10^{-11}$

GREAT QUESTION!

In scientific notation, which numbers have positive exponents and which have negative exponents?

If the absolute value of a number is greater than 10, it will have a positive exponent in scientific notation. If the absolute value of a number is less than 1, it will have a negative exponent in scientific notation.

Check Point 9 Write each number in scientific notation:

a. 5,210,000,000 **b.** −0.00000006893.

EXAMPLE 10 Expressing the U.S. Population in Scientific Notation

As of January 2016, the population of the United States was approximately 322 million. Express the population in scientific notation.

SOLUTION

Because a million is 10^6, the 2016 population can be expressed as

$$322 \times 10^6.$$

This factor is not between 1 and 10, so the number is not in scientific notation.

The voice balloon indicates that we need to convert 322 to scientific notation.

$$322 \times 10^6 = (3.22 \times 10^2) \times 10^6 = 3.22 \times 10^{2+6} = 3.22 \times 10^8$$

$322 = 3.22 \times 10^2$

In scientific notation, the population is 3.22×10^8.

Check Point 10 Express 410×10^7 in scientific notation.

Computations with Scientific Notation

Properties of exponents are used to perform computations with numbers that are expressed in scientific notation.

TECHNOLOGY

$(6.1 \times 10^5)(4 \times 10^{-9})$

On a Calculator:

Many Scientific Calculators

6.1 $\boxed{\text{EE}}$ 5 $\boxed{\times}$ 4 $\boxed{\text{EE}}$ 9 $\boxed{+/-}$ $\boxed{=}$

Display

2.44 − 03

Many Graphing Calculators

6.1 $\boxed{\text{EE}}$ 5 $\boxed{\times}$ 4 $\boxed{\text{EE}}$ $\boxed{(-)}$ 9 $\boxed{\text{ENTER}}$

Display (in scientific notation mode)

2.44E − 3

EXAMPLE 11 Computations with Scientific Notation

Perform the indicated computations, writing the answers in scientific notation:

a. $(6.1 \times 10^5)(4 \times 10^{-9})$ **b.** $\dfrac{1.8 \times 10^4}{3 \times 10^{-2}}$.

SOLUTION

a. $(6.1 \times 10^5)(4 \times 10^{-9})$

$= (6.1 \times 4) \times (10^5 \times 10^{-9})$ Regroup factors.

$= 24.4 \times 10^{5+(-9)}$ Add the exponents on 10 and multiply the other parts.

$= 24.4 \times 10^{-4}$ Simplify.

$= (2.44 \times 10^1) \times 10^{-4}$ Convert 24.4 to scientific notation: $24.4 = 2.44 \times 10^1$.

$= 2.44 \times 10^{-3}$ $10^1 \times 10^{-4} = 10^{1+(-4)} = 10^{-3}$

b. $\frac{1.8 \times 10^4}{3 \times 10^{-2}} = \left(\frac{1.8}{3}\right) \times \left(\frac{10^4}{10^{-2}}\right)$ Regroup factors.

$= 0.6 \times 10^{4-(-2)}$ Subtract the exponents on 10 and divide the other parts.

$= 0.6 \times 10^6$ Simplify: $4 - (-2) = 4 + 2 = 6$.

$= (6 \times 10^{-1}) \times 10^6$ Convert 0.6 to scientific notation: $0.6 = 6 \times 10^{-1}$.

$= 6 \times 10^5$ $10^{-1} \times 10^6 = 10^{-1+6} = 10^5$

Check Point 11 Perform the indicated computations, writing the answers in scientific notation:

a. $(7.1 \times 10^5)(5 \times 10^{-7})$ **b.** $\frac{1.2 \times 10^6}{3 \times 10^{-3}}$.

Applications: Putting Numbers in Perspective

Due to tax cuts and spending increases, the United States began accumulating large deficits in the 1980s. To finance the deficit, the government had borrowed \$18.9 trillion as of January 2016. The graph in **Figure P.10** shows the national debt increasing over time.

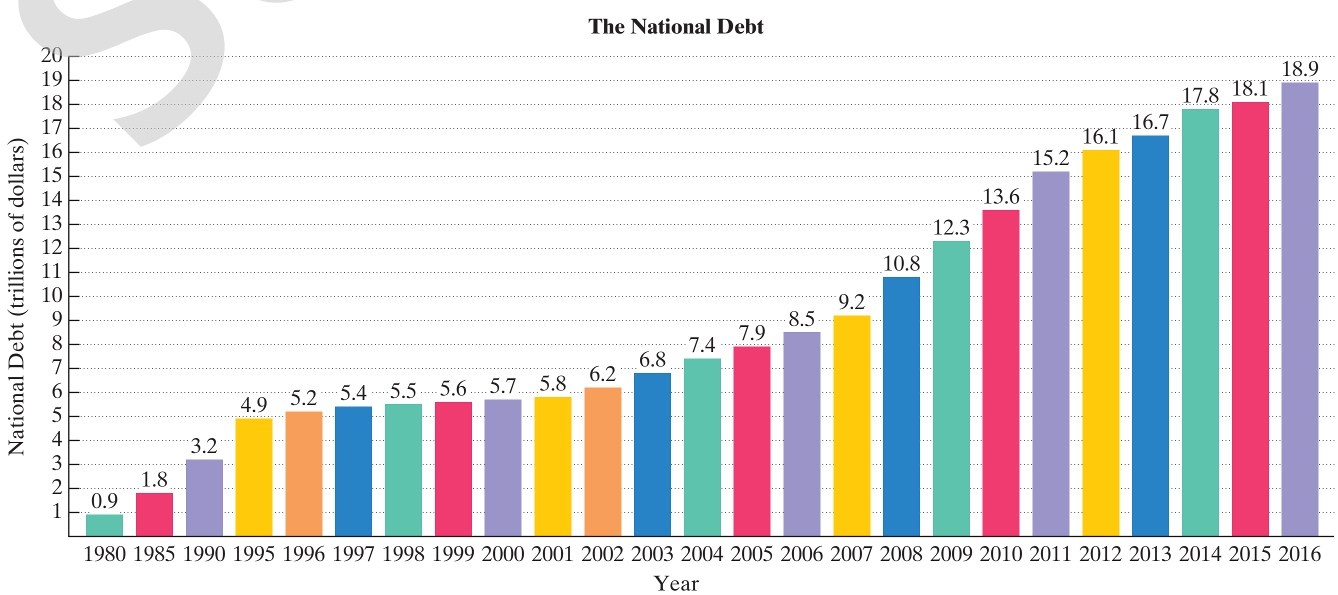


FIGURE P.10
Source: Office of Management and Budget

Example 12 shows how we can use scientific notation to comprehend the meaning of a number such as 18.9 trillion.

EXAMPLE 12 The National Debt

As of January 2016, the national debt was \$18.9 trillion, or 18.9×10^{12} dollars. At that time, the U.S. population was approximately 322,000,000 (322 million), or 3.22×10^8. If the national debt was evenly divided among every individual in the United States, how much would each citizen have to pay?

TECHNOLOGY

Here is the keystroke sequence for solving Example 12 using a calculator:

18.9 $\boxed{\text{EE}}$ 12 $\boxed{\div}$ 3.22 $\boxed{\text{EE}}$ 8.

The quotient is displayed by pressing $\boxed{=}$ on a scientific calculator or $\boxed{\text{ENTER}}$ on a graphing calculator. The answer can be displayed in scientific or decimal notation. Consult your manual.

SOLUTION

The amount each citizen must pay is the total debt, 18.9×10^{12} dollars, divided by the number of citizens, 3.22×10^8.

$$\frac{18.9 \times 10^{12}}{3.22 \times 10^8} = \left(\frac{18.9}{3.22}\right) \times \left(\frac{10^{12}}{10^8}\right)$$
$$\approx 5.87 \times 10^{12-8}$$
$$= 5.87 \times 10^4$$
$$= 58{,}700$$

Every U.S. citizen would have to pay approximately \$58,700 to the federal government to pay off the national debt. • • •

Check Point 12 In 2015, there were 680,000 police officers in the United States with yearly wages totaling $\$4.08 \times 10^{10}$. If these wages were evenly divided among all police officers, find the mean, or average, salary of a U.S. police officer. (Source: Bureau of Justice Statistics)

Blitzer Bonus || Seven Ways to Spend \$1 Trillion

Image © photobank.kiev.ua, 2009

Confronting a national debt of \$18.9 trillion starts with grasping just how colossal \$1 trillion (1×10^{12}) actually is. To help you wrap your head around this mind-boggling number, and to put the national debt in further perspective, consider what \$1 trillion will buy:

- 40,816,326 new cars based on an average sticker price of \$24,500 each
- 5,574,136 homes based on the national median price of \$179,400 for existing single-family homes
- one year's salary for 14.7 million teachers based on the average teacher salary of \$68,000 in California
- the annual salaries of all 535 members of Congress for the next 10,742 years based on current salaries of \$174,000 per year
- the salary of basketball superstar LeBron James for 50,000 years based on an annual salary of \$20 million
- annual base pay for 59.5 million U.S. privates (that's 100 times the total number of active-duty soldiers in the Army) based on basic pay of \$16,794 per year
- salaries to hire all 2.8 million residents of the state of Kansas in full-time minimum-wage jobs for the next 23 years based on the federal minimum wage of \$7.25 per hour

Source: Kiplinger.com

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. The product rule for exponents states that $b^m \cdot b^n =$ ______. When multiplying exponential expressions with the same base, ______ the exponents.
2. The quotient rule for exponents states that $\frac{b^m}{b^n} =$ ______, $b \neq 0$. When dividing exponential expressions with the same nonzero base, ______ the exponents.
3. If $b \neq 0$, then $b^0 =$ ______.
4. The negative-exponent rule states that $b^{-n} =$ ______, $b \neq 0$.
5. True or false: $5^{-2} = -5^2$ ______
6. Negative exponents in denominators can be evaluated using $\frac{1}{b^{-n}} =$ ______, $b \neq 0$.
7. True or false: $\frac{1}{8^{-2}} = 8^2$ ____
8. A positive number is written in scientific notation when it is expressed in the form $a \times 10^n$, where a is ______________________________ and n is a/an ______.
9. True or false: 4×10^3 is written in scientific notation. ____
10. True or false: 40×10^2 is written in scientific notation. ______

EXERCISE SET P.2

Practice Exercises

Evaluate each exponential expression in Exercises 1–22.

1. $5^2 \cdot 2$
2. $6^2 \cdot 2$
3. $(-2)^6$
4. $(-2)^4$
5. -2^6
6. -2^4
7. $(-3)^0$
8. $(-9)^0$
9. -3^0
10. -9^0
11. 4^{-3}
12. 2^{-6}
13. $2^2 \cdot 2^3$
14. $3^3 \cdot 3^2$
15. $(2^2)^3$
16. $(3^3)^2$
17. $\frac{2^8}{2^4}$
18. $\frac{3^8}{3^4}$
19. $3^{-3} \cdot 3$
20. $2^{-3} \cdot 2$
21. $\frac{2^3}{2^7}$
22. $\frac{3^4}{3^7}$

Simplify each exponential expression in Exercises 23–64.

23. $x^{-2}y$
24. xy^{-3}
25. x^0y^5
26. x^7y^0
27. $x^3 \cdot x^7$
28. $x^{11} \cdot x^5$
29. $x^{-5} \cdot x^{10}$
30. $x^{-6} \cdot x^{12}$
31. $(x^3)^7$
32. $(x^{11})^5$
33. $(x^{-5})^3$
34. $(x^{-6})^4$
35. $\frac{x^{14}}{x^7}$
36. $\frac{x^{30}}{x^{10}}$
37. $\frac{x^{14}}{x^{-7}}$
38. $\frac{x^{30}}{x^{-10}}$
39. $(8x^3)^2$
40. $(6x^4)^2$
41. $\left(-\frac{4}{x}\right)^3$
42. $\left(-\frac{6}{y}\right)^3$
43. $(-3x^2y^5)^2$
44. $(-3x^4y^6)^3$
45. $(3x^4)(2x^7)$
46. $(11x^5)(9x^{12})$
47. $(-9x^3y)(-2x^6y^4)$
48. $(-5x^4y)(-6x^7y^{11})$
49. $\frac{8x^{20}}{2x^4}$
50. $\frac{20x^{24}}{10x^6}$
51. $\frac{25a^{13}b^4}{-5a^2b^3}$
52. $\frac{35a^{14}b^6}{-7a^7b^3}$
53. $\frac{14b^7}{7b^{14}}$
54. $\frac{20b^{10}}{10b^{20}}$
55. $(4x^3)^{-2}$
56. $(10x^2)^{-3}$
57. $\frac{24x^3y^5}{32x^7y^{-9}}$
58. $\frac{10x^4y^9}{30x^{12}y^{-3}}$
59. $\left(\frac{5x^3}{y}\right)^{-2}$
60. $\left(\frac{3x^4}{y}\right)^{-3}$
61. $\left(\frac{-15a^4b^2}{5a^{10}b^{-3}}\right)^3$
62. $\left(\frac{-30a^{14}b^8}{10a^{17}b^{-2}}\right)^3$
63. $\left(\frac{3a^{-5}b^2}{12a^3b^{-4}}\right)^0$
64. $\left(\frac{4a^{-5}b^3}{12a^3b^{-5}}\right)^0$

In Exercises 65–76, write each number in decimal notation without the use of exponents.

65. 3.8×10^2
66. 9.2×10^2
67. 6×10^{-4}
68. 7×10^{-5}
69. -7.16×10^6
70. -8.17×10^6
71. 7.9×10^{-1}
72. 6.8×10^{-1}
73. -4.15×10^{-3}
74. -3.14×10^{-3}
75. -6.00001×10^{10}
76. -7.00001×10^{10}

In Exercises 77–86, write each number in scientific notation.

77. 32,000
78. 64,000
79. 638,000,000,000,000,000
80. 579,000,000,000,000,000
81. -5716
82. -3829
83. 0.0027
84. 0.0083
85. -0.00000000504
86. -0.00000000405

In Exercises 87–106, perform the indicated computations. Write the answers in scientific notation. If necessary, round the decimal factor in your scientific notation answer to two decimal places.

87. $(3 \times 10^4)(2.1 \times 10^3)$
88. $(2 \times 10^4)(4.1 \times 10^3)$
89. $(1.6 \times 10^{15})(4 \times 10^{-11})$
90. $(1.4 \times 10^{15})(3 \times 10^{-11})$
91. $(6.1 \times 10^{-8})(2 \times 10^{-4})$
92. $(5.1 \times 10^{-8})(3 \times 10^{-4})$
93. $(4.3 \times 10^8)(6.2 \times 10^4)$
94. $(8.2 \times 10^8)(4.6 \times 10^4)$
95. $\frac{8.4 \times 10^8}{4 \times 10^5}$
96. $\frac{6.9 \times 10^8}{3 \times 10^5}$
97. $\frac{3.6 \times 10^4}{9 \times 10^{-2}}$
98. $\frac{1.2 \times 10^4}{2 \times 10^{-2}}$
99. $\frac{4.8 \times 10^{-2}}{2.4 \times 10^6}$
100. $\frac{7.5 \times 10^{-2}}{2.5 \times 10^6}$
101. $\frac{2.4 \times 10^{-2}}{4.8 \times 10^{-6}}$
102. $\frac{1.5 \times 10^{-2}}{3 \times 10^{-6}}$
103. $\frac{480{,}000{,}000{,}000}{0.00012}$
104. $\frac{282{,}000{,}000{,}000}{0.00141}$
105. $\frac{0.00072 \times 0.003}{0.00024}$
106. $\frac{66{,}000 \times 0.001}{0.003 \times 0.002}$

Practice Plus

In Exercises 107–114, simplify each exponential expression. Assume that variables represent nonzero real numbers.

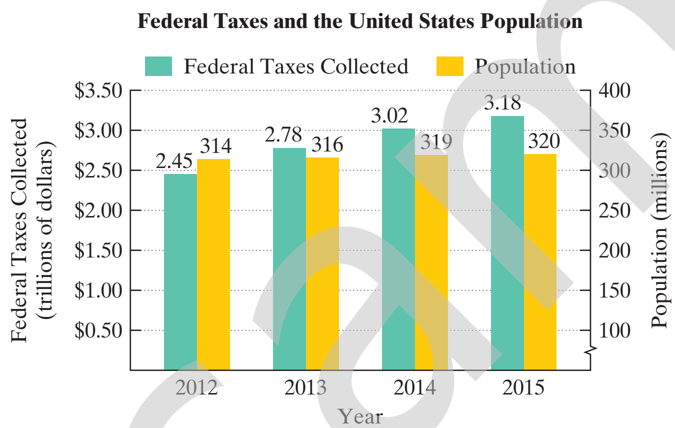
107. $\frac{(x^{-2}y)^{-3}}{(x^2y^{-1})^3}$
108. $\frac{(xy^{-2})^{-2}}{(x^{-2}y)^{-3}}$
109. $(2x^{-3}yz^{-6})(2x)^{-5}$
110. $(3x^{-4}yz^{-7})(3x)^{-3}$
111. $\left(\frac{x^3y^4z^5}{x^{-3}y^{-4}z^{-5}}\right)^{-2}$
112. $\left(\frac{x^4y^5z^6}{x^{-4}y^{-5}z^{-6}}\right)^{-4}$

113. $\dfrac{(2^{-1}x^{-2}y^{-1})^{-2}(2x^{-4}y^{3})^{-2}(16x^{-3}y^{3})^{0}}{(2x^{-3}y^{-5})^{2}}$

114. $\dfrac{(2^{-1}x^{-3}y^{-1})^{-2}(2x^{-6}y^{4})^{-2}(9x^{3}y^{-3})^{0}}{(2x^{-4}y^{-6})^{2}}$

Application Exercises

The bar graph shows the total amount Americans paid in federal taxes, in trillions of dollars, and the U.S. population, in millions, from 2012 through 2015. Exercises 115–116 are based on the numbers displayed by the graph.



Sources: Internal Revenue Service and U.S. Census Bureau

115. a. In 2015, the United States government collected \$3.18 trillion in taxes. Express this number in scientific notation.
   b. In 2015, the population of the United States was approximately 320 million. Express this number in scientific notation.
   c. Use your scientific notation answers from parts (a) and (b) to answer this question: If the total 2015 tax collections were evenly divided among all Americans, how much would each citizen pay? Express the answer in decimal notation, rounded to the nearest dollar.

116. a. In 2014, the United States government collected \$3.02 trillion in taxes. Express this number in scientific notation.
   b. In 2014, the population of the United States was approximately 319 million. Express this number in scientific notation.
   c. Use your scientific notation answers from parts (a) and (b) to answer this question: If the total 2014 tax collections were evenly divided among all Americans, how much would each citizen pay? Express the answer in decimal notation, rounded to the nearest dollar.

We have seen that the 2016 U.S. national debt was \$18.9 trillion. In Exercises 117–118, you will use scientific notation to put a number like 18.9 trillion in perspective.

117. a. Express 18.9 trillion in scientific notation.
   b. Four years of tuition, fees, and room and board at a public U.S. college cost approximately \$60,000. Express this number in scientific notation.
   c. Use your answers from parts (a) and (b) to determine how many Americans could receive a free college education for \$18.9 trillion.

118. a. Express 18.9 trillion in scientific notation.
   b. Each year, Americans spend \$254 billion on summer vacations. Express this number in scientific notation.
   c. Use your answers from parts (a) and (b) to determine how many years Americans can have free summer vacations for \$18.9 trillion.

119. In 2012, the United States government spent more than it had collected in taxes, resulting in a budget deficit of \$1.09 trillion.
   a. Express 1.09 trillion in scientific notation.
   b. There are approximately 32,000,000 seconds in a year. Write this number in scienfic notation.
   c. Use your answers from parts (a) and (b) to determine approximately how many years is 1.09 trillion seconds. (Note: 1.09 trillion seconds would take us back in time to a period when Neanderthals were using stones to make tools.)

120. Refer to the Blitzer Bonus on page 32. Use scientific notation to verify any two of the bulleted items on ways to spend \$1 trillion.

Explaining the Concepts

121. Describe what it means to raise a number to a power. In your description, include a discussion of the difference between -5^2 and $(-5)^2$.

122. Explain the product rule for exponents. Use $2^3 \cdot 2^5$ in your explanation.

123. Explain the power rule for exponents. Use $(3^2)^4$ in your explanation.

124. Explain the quotient rule for exponents. Use $\dfrac{5^8}{5^2}$ in your explanation.

125. Why is $(-3x^2)(2x^{-5})$ not simplified? What must be done to simplify the expression?

126. How do you know if a number is written in scientific notation?

127. Explain how to convert from scientific to decimal notation and give an example.

128. Explain how to convert from decimal to scientific notation and give an example.

Critical Thinking Exercises

Make Sense? *In Exercises 129–132, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

129. There are many exponential expressions that are equal to $36x^{12}$, such as $(6x^6)^2$, $(6x^3)(6x^9)$, $36(x^3)^9$, and $6^2(x^2)^6$.

130. If 5^{-2} is raised to the third power, the result is a number between 0 and 1.

131. The population of Colorado is approximately 4.6×10^{12}.

132. I just finished reading a book that contained approximately 1.04×10^5 words.

In Exercises 133–140, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

133. $4^{-2} < 4^{-3}$

134. $5^{-2} > 2^{-5}$

135. $(-2)^4 = 2^{-4}$

136. $5^2 \cdot 5^{-2} > 2^5 \cdot 2^{-5}$

137. $534.7 = 5.347 \times 10^3$

138. $\dfrac{8 \times 10^{30}}{4 \times 10^{-5}} = 2 \times 10^{25}$

139. $(7 \times 10^5) + (2 \times 10^{-3}) = 9 \times 10^2$

140. $(4 \times 10^3) + (3 \times 10^2) = 4.3 \times 10^3$

141. The mad Dr. Frankenstein has gathered enough bits and pieces (so to speak) for $2^{-1} + 2^{-2}$ of his creature-to-be. Write a fraction that represents the amount of his creature that must still be obtained.

142. If $b^A = MN$, $b^C = M$, and $b^D = N$, what is the relationship among A, C, and D?

143. Our hearts beat approximately 70 times per minute. Express in scientific notation how many times the heart beats over a lifetime of 80 years. Round the decimal factor in your scientific notation answer to two decimal places.

Group Exercise

144. **Putting Numbers into Perspective.** A large number can be put into perspective by comparing it with another number. For example, we put the \$18.9 trillion national debt in perspective (Example 12) by comparing this number to the number of U.S. citizens.

For this project, each group member should consult an almanac, a newspaper, or the Internet to find a number greater than one million. Explain to other members of the group the context in which the large number is used. Express the number in scientific notation. Then put the number into perspective by comparing it with another number.

Preview Exercises

Exercises 145–147 will help you prepare for the material covered in the next section.

145. **a.** Find $\sqrt{16} \cdot \sqrt{4}$.

b. Find $\sqrt{16 \cdot 4}$.

c. Based on your answers to parts (a) and (b), what can you conclude?

146. **a.** Use a calculator to approximate $\sqrt{300}$ to two decimal places.

b. Use a calculator to approximate $10\sqrt{3}$ to two decimal places.

c. Based on your answers to parts (a) and (b), what can you conclude?

147. **a.** Simplify: $21x + 10x$.

b. Simplify: $21\sqrt{2} + 10\sqrt{2}$.

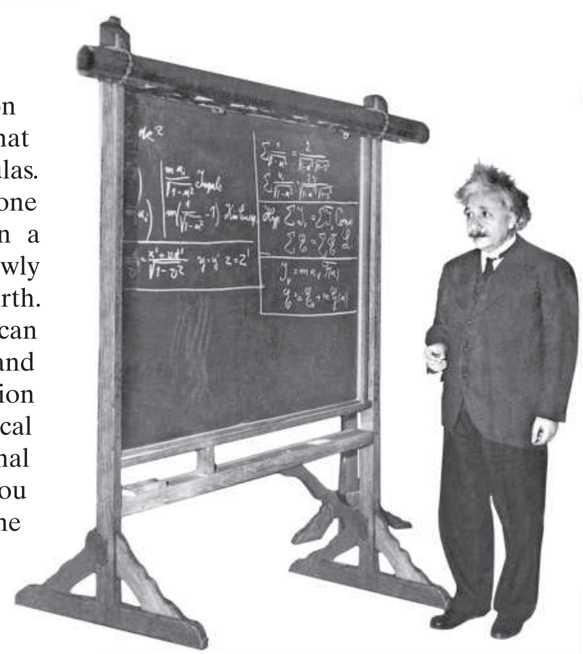
Section P.3 Radicals and Rational Exponents

What am I supposed to learn?

After studying this section, you should be able to:

1. Evaluate square roots.
2. Simplify expressions of the form $\sqrt{a^2}$.
3. Use the product rule to simplify square roots.
4. Use the quotient rule to simplify square roots.
5. Add and subtract square roots.
6. Rationalize denominators.
7. Evaluate and perform operations with higher roots.
8. Understand and use rational exponents.

This photograph shows mathematical models used by Albert Einstein at a lecture on relativity. Notice the radicals that appear in many of the formulas. Among these models, there is one describing how an astronaut in a moving spaceship ages more slowly than friends who remain on Earth. No description of your world can be complete without roots and radicals. In this section, in addition to reviewing the basics of radical expressions and the use of rational exponents to indicate radicals, you will see how radicals model time dilation for a futuristic high-speed trip to a nearby star.



1 Evaluate square roots.

Square Roots

From our earlier work with exponents, we are aware that the square of both 5 and -5 is 25:

$$5^2 = 25 \quad \text{and} \quad (-5)^2 = 25.$$

The reverse operation of squaring a number is finding the *square root* of the number. For example,

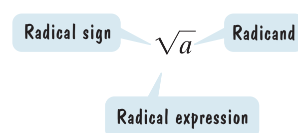
- One square root of 25 is 5 because $5^2 = 25$.
- Another square root of 25 is -5 because $(-5)^2 = 25$.

In general, **if $b^2 = a$, then b is a square root of a.**

The symbol $\sqrt{\ }$ is used to denote the *nonnegative* or *principal square root* of a number. For example,

- $\sqrt{25} = 5$ because $5^2 = 25$ and 5 is positive.
- $\sqrt{100} = 10$ because $10^2 = 100$ and 10 is positive.

The symbol $\sqrt{\ }$ that we use to denote the principal square root is called a **radical sign**. The number under the radical sign is called the **radicand**. Together we refer to the radical sign and its radicand as a **radical expression**.



Definition of the Principal Square Root

If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$, denoted by $b = \sqrt{a}$, is the **principal square root** of a.

The symbol $-\sqrt{\ }$ is used to denote the negative square root of a number. For example,

- $-\sqrt{25} = -5$ because $(-5)^2 = 25$ and -5 is negative.
- $-\sqrt{100} = -10$ because $(-10)^2 = 100$ and -10 is negative.

EXAMPLE 1 Evaluating Square Roots

Evaluate:

a. $\sqrt{64}$ **b.** $-\sqrt{49}$ **c.** $\sqrt{\frac{1}{4}}$ **d.** $\sqrt{9+16}$ **e.** $\sqrt{9} + \sqrt{16}$.

SOLUTION

a. $\sqrt{64} = 8$ — The principal square root of 64 is 8. Check: $8^2 = 64$.

b. $-\sqrt{49} = -7$ — The negative square root of 49 is -7. Check: $(-7)^2 = 49$.

c. $\sqrt{\frac{1}{4}} = \frac{1}{2}$ — The principal square root of $\frac{1}{4}$ is $\frac{1}{2}$. Check: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

d. $\sqrt{9+16} = \sqrt{25}$ — First simplify the expression under the radical sign.

$= 5$ — Then take the principal square root of 25, which is 5.

e. $\sqrt{9} + \sqrt{16} = 3 + 4$ — $\sqrt{9} = 3$ because $3^2 = 9$. $\sqrt{16} = 4$ because $4^2 = 16$.

$= 7$

GREAT QUESTION!

Is $\sqrt{a+b}$ equal to $\sqrt{a} + \sqrt{b}$?

No. In Example 1, parts (d) and (e), observe that $\sqrt{9+16}$ is not equal to $\sqrt{9} + \sqrt{16}$. In general,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

and

$$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}.$$

Check Point 1 Evaluate:

a. $\sqrt{81}$ **b.** $-\sqrt{9}$ **c.** $\sqrt{\frac{1}{25}}$

d. $\sqrt{36 + 64}$ **e.** $\sqrt{36} + \sqrt{64}$.

A number that is the square of a rational number is called a **perfect square**. All the radicands in Example 1 and Check Point 1 are perfect squares.

- 64 is a perfect square because $64 = 8^2$. Thus, $\sqrt{64} = 8$.
- $\frac{1}{4}$ is a perfect square because $\frac{1}{4} = \left(\frac{1}{2}\right)^2$. Thus, $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Let's see what happens to the radical expression $\sqrt{x}$ if x is a negative number. Is the square root of a negative number a real number? For example, consider $\sqrt{-25}$. Is there a real number whose square is -25? No. Thus, $\sqrt{-25}$ is not a real number. In general, **a square root of a negative number is not a real number.**

If a number a is nonnegative $(a \geq 0)$, then $(\sqrt{a})^2 = a$. For example,

$$(\sqrt{2})^2 = 2, \quad (\sqrt{3})^2 = 3, \quad (\sqrt{4})^2 = 4, \quad \text{and} \quad (\sqrt{5})^2 = 5.$$

2 Simplify expressions of the form $\sqrt{a^2}$.

Simplifying Expressions of the Form $\sqrt{a^2}$

You may think that $\sqrt{a^2} = a$. However, this is not necessarily true. Consider the following examples:

$$\sqrt{4^2} = \sqrt{16} = 4$$
$$\sqrt{(-4)^2} = \sqrt{16} = 4.$$

The result is not -4, but rather the absolute value of -4, or 4.

Here is a rule for simplifying expressions of the form $\sqrt{a^2}$:

Simplifying $\sqrt{a^2}$

For any real number a,

$$\sqrt{a^2} = |a|.$$

In words, the principal square root of a^2 is the absolute value of a.

For example, $\sqrt{6^2} = |6| = 6$ and $\sqrt{(-6)^2} = |-6| = 6$.

3 Use the product rule to simplify square roots.

The Product Rule for Square Roots

A rule for multiplying square roots can be generalized by comparing $\sqrt{25} \cdot \sqrt{4}$ and $\sqrt{25 \cdot 4}$. Notice that

$$\sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10 \quad \text{and} \quad \sqrt{25 \cdot 4} = \sqrt{100} = 10.$$

Because we obtain 10 in both situations, the original radical expressions must be equal. That is,

$$\sqrt{25} \cdot \sqrt{4} = \sqrt{25 \cdot 4}.$$

This result is a special case of the **product rule for square roots** that can be generalized as follows:

The Product Rule for Square Roots

If a and b represent nonnegative real numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

The square root of a product is the product of the square roots.

The product of two square roots is the square root of the product of the radicands.

A square root is **simplified** when its radicand has no factors other than 1 that are perfect squares. For example, $\sqrt{500}$ is not simplified because it can be expressed as $\sqrt{100 \cdot 5}$ and 100 is a perfect square. Example 2 shows how the product rule is used to remove from the square root any perfect squares that occur as factors.

EXAMPLE 2 Using the Product Rule to Simplify Square Roots

Simplify:

a. $\sqrt{500}$ **b.** $\sqrt{6x} \cdot \sqrt{3x}$.

SOLUTION

a.
$$\sqrt{500} = \sqrt{100 \cdot 5} \quad \text{Factor 500. 100 is the greatest perfect square factor.}$$
$$= \sqrt{100}\sqrt{5} \quad \text{Use the product rule: } \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$
$$= 10\sqrt{5} \quad \text{Write } \sqrt{100} \text{ as 10. We read } 10\sqrt{5} \text{ as "ten times the square root of 5."}$$

b. We can simplify $\sqrt{6x} \cdot \sqrt{3x}$ using the product rule only if $6x$ and $3x$ represent nonnegative real numbers. Thus, $x \geq 0$.

$$\sqrt{6x} \cdot \sqrt{3x} = \sqrt{6x \cdot 3x} \quad \text{Use the product rule: } \sqrt{a}\sqrt{b} = \sqrt{ab}.$$
$$= \sqrt{18x^2} \quad \text{Multiply in the radicand.}$$
$$= \sqrt{9x^2 \cdot 2} \quad \text{Factor 18. 9 is the greatest perfect square factor.}$$
$$= \sqrt{9x^2}\sqrt{2} \quad \text{Use the product rule: } \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$
$$= \sqrt{9}\sqrt{x^2}\sqrt{2} \quad \text{Use the product rule to write } \sqrt{9x^2} \text{ as the product of two square roots.}$$
$$= 3x\sqrt{2} \quad \sqrt{x^2} = |x| = x \text{ because } x \geq 0.$$

GREAT QUESTION!

When simplifying square roots, what happens if I use a perfect square factor that isn't the greatest perfect square factor possible?

You'll need to simplify even further. For example, consider the following factorization:

$$\sqrt{500} = \sqrt{25 \cdot 20} = \sqrt{25}\sqrt{20} = 5\sqrt{20}.$$

25 is a perfect square factor of 500, but not the greatest perfect square factor.

Because 20 contains a perfect square factor, 4, the simplification is not complete.

$$5\sqrt{20} = 5\sqrt{4 \cdot 5} = 5\sqrt{4}\sqrt{5} = 5 \cdot 2\sqrt{5} = 10\sqrt{5}$$

Although the result checks with our simplification using $\sqrt{500} = \sqrt{100 \cdot 5}$, **more work is required when the greatest perfect square factor is not used**.

Check Point 2 Simplify:

a. $\sqrt{75}$ **b.** $\sqrt{5x} \cdot \sqrt{10x}$.

4 Use the quotient rule to simplify square roots.

The Quotient Rule for Square Roots

Another property for square roots involves division.

The Quotient Rule for Square Roots

If a and b represent nonnegative real numbers and $b \neq 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

The square root of a quotient is the quotient of the square roots.

The quotient of two square roots is the square root of the quotient of the radicands.

EXAMPLE 3 Using the Quotient Rule to Simplify Square Roots

Simplify:

a. $\sqrt{\dfrac{100}{9}}$ **b.** $\dfrac{\sqrt{48x^3}}{\sqrt{6x}}$.

SOLUTION

a. $\sqrt{\dfrac{100}{9}} = \dfrac{\sqrt{100}}{\sqrt{9}} = \dfrac{10}{3}$

b. We can simplify the quotient of $\sqrt{48x^3}$ and $\sqrt{6x}$ using the quotient rule only if $48x^3$ and $6x$ represent nonnegative real numbers and $6x \neq 0$. Thus, $x > 0$.

$$\frac{\sqrt{48x^3}}{\sqrt{6x}} = \sqrt{\frac{48x^3}{6x}} = \sqrt{8x^2} = \sqrt{4x^2}\sqrt{2} = \sqrt{4}\sqrt{x^2}\sqrt{2} = 2x\sqrt{2}$$

$\sqrt{x^2} = |x| = x$ because $x > 0$.

Check Point 3 Simplify:

a. $\sqrt{\dfrac{25}{16}}$ **b.** $\dfrac{\sqrt{150x^3}}{\sqrt{2x}}$.

5 Add and subtract square roots.

Adding and Subtracting Square Roots

Two or more square roots can be combined using the distributive property provided that they have the same radicand. Such radicals are called **like radicals**. For example,

$$7\sqrt{11} + 6\sqrt{11} = (7 + 6)\sqrt{11} = 13\sqrt{11}.$$

7 square roots of 11 plus 6 square roots of 11 result in 13 square roots of 11.

GREAT QUESTION!

Should like radicals remind me of like terms?

Yes. Adding or subtracting like radicals is similar to adding or subtracting like terms:

$$7x + 6x = 13x$$

and

$$7\sqrt{11} + 6\sqrt{11} = 13\sqrt{11}.$$

EXAMPLE 4 Adding and Subtracting Like Radicals

Add or subtract as indicated:

a. $7\sqrt{2} + 5\sqrt{2}$ **b.** $\sqrt{5x} - 7\sqrt{5x}$.

SOLUTION

a. $7\sqrt{2} + 5\sqrt{2} = (7 + 5)\sqrt{2}$ Apply the distributive property.

$= 12\sqrt{2}$ Simplify.

b. $\sqrt{5x} - 7\sqrt{5x} = 1\sqrt{5x} - 7\sqrt{5x}$ Write $\sqrt{5x}$ as $1\sqrt{5x}$.

$= (1 - 7)\sqrt{5x}$ Apply the distributive property.

$= -6\sqrt{5x}$ Simplify.

Check Point 4 Add or subtract as indicated:

a. $8\sqrt{13} + 9\sqrt{13}$ **b.** $\sqrt{17x} - 20\sqrt{17x}$.

In some cases, radicals can be combined once they have been simplified. For example, to add $\sqrt{2}$ and $\sqrt{8}$, we can write $\sqrt{8}$ as $\sqrt{4 \cdot 2}$ because 4 is a perfect square factor of 8.

$$\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4 \cdot 2} = 1\sqrt{2} + 2\sqrt{2} = (1 + 2)\sqrt{2} = 3\sqrt{2}$$

EXAMPLE 5 Combining Radicals That First Require Simplification

Add or subtract as indicated:

a. $7\sqrt{3} + \sqrt{12}$ **b.** $4\sqrt{50x} - 6\sqrt{32x}$.

SOLUTION

a. $7\sqrt{3} + \sqrt{12}$

$= 7\sqrt{3} + \sqrt{4 \cdot 3}$ — Split 12 into two factors such that one is a perfect square.

$= 7\sqrt{3} + 2\sqrt{3}$ — $\sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

$= (7 + 2)\sqrt{3}$ — Apply the distributive property. You will find that this step is usually done mentally.

$= 9\sqrt{3}$ — Simplify.

b. $4\sqrt{50x} - 6\sqrt{32x}$

$= 4\sqrt{25 \cdot 2x} - 6\sqrt{16 \cdot 2x}$ — 25 is the greatest perfect square factor of $50x$ and 16 is the greatest perfect square factor of $32x$.

$= 4 \cdot 5\sqrt{2x} - 6 \cdot 4\sqrt{2x}$ — $\sqrt{25 \cdot 2x} = \sqrt{25}\sqrt{2x} = 5\sqrt{2x}$ and $\sqrt{16 \cdot 2x} = \sqrt{16}\sqrt{2x} = 4\sqrt{2x}$.

$= 20\sqrt{2x} - 24\sqrt{2x}$ — Multiply: $4 \cdot 5 = 20$ and $6 \cdot 4 = 24$.

$= (20 - 24)\sqrt{2x}$ — Apply the distributive property.

$= -4\sqrt{2x}$ — Simplify.

Check Point 5 Add or subtract as indicated:

a. $5\sqrt{27} + \sqrt{12}$ **b.** $6\sqrt{18x} - 4\sqrt{8x}$.

6 Rationalize denominators.

Rationalizing Denominators

The calculator screen in **Figure P.11** shows approximate values for $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$. The two approximations are the same. This is not a coincidence:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \boxed{\frac{\sqrt{3}}{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}.$$

Any number divided by itself is 1. Multiplication by 1 does not change the value of $\frac{1}{\sqrt{3}}$.

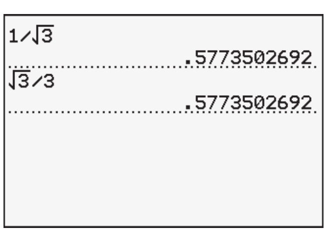


FIGURE P.11 The calculator screen shows approximate values for $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$.

This process involves rewriting a radical expression as an equivalent expression in which the denominator no longer contains any radicals. The process is called **rationalizing the denominator**. If the denominator consists of the square root of a natural number that is not a perfect square, **multiply the numerator and the denominator by the smallest number that produces the square root of a perfect square in the denominator**.

EXAMPLE 6 Rationalizing Denominators

Rationalize the denominator:

a. $\frac{15}{\sqrt{6}}$ **b.** $\frac{12}{\sqrt{8}}$.

GREAT QUESTION!

What exactly does rationalizing a denominator do to an irrational number in the denominator?

Rationalizing a numerical denominator makes that denominator a rational number.

SOLUTION

a. If we multiply the numerator and the denominator of $\frac{15}{\sqrt{6}}$ by $\sqrt{6}$, the denominator becomes $\sqrt{6}\cdot\sqrt{6} = \sqrt{36} = 6$. Therefore, we multiply by 1, choosing $\frac{\sqrt{6}}{\sqrt{6}}$ for 1.

$$\frac{15}{\sqrt{6}} = \frac{15}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = \frac{15\sqrt{6}}{\sqrt{36}} = \frac{15\sqrt{6}}{6} = \frac{5\sqrt{6}}{2}$$

Multiply by 1.

Simplify: $\frac{15}{6} = \frac{15 \div 3}{6 \div 3} = \frac{5}{2}$.

b. The *smallest* number that will produce the square root of a perfect square in the denominator of $\frac{12}{\sqrt{8}}$ is $\sqrt{2}$, because $\sqrt{8}\cdot\sqrt{2} = \sqrt{16} = 4$. We multiply by 1, choosing $\frac{\sqrt{2}}{\sqrt{2}}$ for 1.

$$\frac{12}{\sqrt{8}} = \frac{12}{\sqrt{8}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{\sqrt{16}} = \frac{12\sqrt{2}}{4} = 3\sqrt{2}$$

Check Point 6 Rationalize the denominator:

a. $\frac{5}{\sqrt{3}}$ **b.** $\frac{6}{\sqrt{12}}$.

Radical expressions that involve the sum and difference of the same two terms are called **conjugates**. Thus,

$$\sqrt{a} + \sqrt{b} \quad \text{and} \quad \sqrt{a} - \sqrt{b}$$

are conjugates. Conjugates are used to rationalize denominators because the product of such a pair contains no radicals:

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$$

Multiply each term of $\sqrt{a} - \sqrt{b}$ by each term of $\sqrt{a} + \sqrt{b}$.

$$= \sqrt{a}(\sqrt{a} - \sqrt{b}) + \sqrt{b}(\sqrt{a} - \sqrt{b})$$

Distribute $\sqrt{a}$ over $\sqrt{a} - \sqrt{b}$.

Distribute $\sqrt{b}$ over $\sqrt{a} - \sqrt{b}$.

$$= \sqrt{a}\cdot\sqrt{a} - \sqrt{a}\cdot\sqrt{b} + \sqrt{b}\cdot\sqrt{a} - \sqrt{b}\cdot\sqrt{b}$$

$$= (\sqrt{a})^2 - \sqrt{ab} + \sqrt{ab} - (\sqrt{b})^2$$

$-\sqrt{ab} + \sqrt{ab} = 0$

$$= (\sqrt{a})^2 - (\sqrt{b})^2$$

$$= a - b.$$

Multiplying Conjugates

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$$

How can we rationalize a denominator if the denominator contains two terms with one or more square roots? **Multiply the numerator and the denominator by the conjugate of the denominator.** Here are three examples of such expressions:

- $\dfrac{7}{5+\sqrt{3}}$ The conjugate of the denominator is $5-\sqrt{3}$.
- $\dfrac{8}{3\sqrt{2}-4}$ The conjugate of the denominator is $3\sqrt{2}+4$.
- $\dfrac{h}{\sqrt{x+h}-\sqrt{x}}$ The conjugate of the denominator is $\sqrt{x+h}+\sqrt{x}$.

The product of the denominator and its conjugate is found using the formula

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b.$$

The simplified product will not contain a radical.

EXAMPLE 7 Rationalizing a Denominator Containing Two Terms

Rationalize the denominator: $\dfrac{7}{5+\sqrt{3}}$.

SOLUTION

The conjugate of the denominator is $5-\sqrt{3}$. If we multiply the numerator and denominator by $5-\sqrt{3}$, the simplified denominator will not contain a radical. Therefore, we multiply by 1, choosing $\dfrac{5-\sqrt{3}}{5-\sqrt{3}}$ for 1.

$$\frac{7}{5+\sqrt{3}} = \underbrace{\frac{7}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}}}_{\text{Multiply by 1.}} = \underbrace{\frac{7(5-\sqrt{3})}{5^2-(\sqrt{3})^2}}_{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2} = \frac{7(5-\sqrt{3})}{25-3}$$

$$= \frac{7(5-\sqrt{3})}{22} \quad \text{or} \quad \frac{35-7\sqrt{3}}{22}$$

In either form of the answer, there is no radical in the denominator.

Check Point 7 Rationalize the denominator: $\dfrac{8}{4+\sqrt{5}}$.

7 Evaluate and perform operations with higher roots.

Other Kinds of Roots

We define the **principal *n*th root** of a real number a, symbolized by $\sqrt[n]{a}$, as follows:

Definition of the Principal *n*th Root of a Real Number

$$\sqrt[n]{a} = b \text{ means that } b^n = a.$$

If n, the **index**, is even, then a is nonnegative ($a \geq 0$) and b is also nonnegative ($b \geq 0$). If n is odd, a and b can be any real numbers.

For example,

$$\sqrt[3]{64} = 4 \text{ because } 4^3 = 64 \quad \text{and} \quad \sqrt[5]{-32} = -2 \text{ because } (-2)^5 = -32.$$

The same vocabulary that we learned for square roots applies to nth roots. The symbol $\sqrt[n]{}$ is called a **radical** and the expression under the radical is called the **radicand**.

GREAT QUESTION!

Should I know the higher roots of certain numbers by heart?

Some higher roots occur so frequently that you might want to memorize them.

Cube Roots	
$\sqrt[3]{1} = 1$	$\sqrt[3]{125} = 5$
$\sqrt[3]{8} = 2$	$\sqrt[3]{216} = 6$
$\sqrt[3]{27} = 3$	$\sqrt[3]{1000} = 10$
$\sqrt[3]{64} = 4$	

Fourth Roots
$\sqrt[4]{1} = 1$
$\sqrt[4]{16} = 2$
$\sqrt[4]{81} = 3$
$\sqrt[4]{256} = 4$
$\sqrt[4]{625} = 5$

Fifth Roots
$\sqrt[5]{1} = 1$
$\sqrt[5]{32} = 2$
$\sqrt[5]{243} = 3$

A number that is the nth power of a rational number is called a **perfect nth power**. For example, 8 is a perfect third power, or perfect cube, because $8 = 2^3$. Thus, $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$. In general, one of the following rules can be used to find the nth root of a perfect nth power:

Finding nth Roots of Perfect nth Powers

If n is odd, $\sqrt[n]{a^n} = a$.

If n is even, $\sqrt[n]{a^n} = |a|$.

For example,

$$\sqrt[3]{(-2)^3} = -2 \qquad \text{and} \qquad \sqrt[4]{(-2)^4} = |-2| = 2.$$

Absolute value is not needed with odd roots, but is necessary with even roots.

The Product and Quotient Rules for Other Roots

The product and quotient rules apply to cube roots, fourth roots, and all higher roots.

The Product and Quotient Rules for nth Roots

For all real numbers a and b, where the indicated roots represent real numbers,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad \text{and} \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

The nth root of a product is the product of the nth roots.

The product of two nth roots is the nth root of the product of the radicands.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},\ b \neq 0 \qquad \text{and} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},\ b \neq 0.$$

The nth root of a quotient is the quotient of the nth roots.

The quotient of two nth roots is the nth root of the quotient of the radicands.

EXAMPLE 8 Simplifying, Multiplying, and Dividing Higher Roots

Simplify: **a.** $\sqrt[3]{24}$ **b.** $\sqrt[4]{8} \cdot \sqrt[4]{4}$ **c.** $\sqrt[4]{\frac{81}{16}}$.

SOLUTION

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3}$ Find the greatest perfect cube that is a factor of 24. $2^3 = 8$, so 8 is a perfect cube and is the greatest perfect cube factor of 24.

$= \sqrt[3]{8} \cdot \sqrt[3]{3}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 2\sqrt[3]{3}$ $\sqrt[3]{8} = 2$

b. $\sqrt[4]{8} \cdot \sqrt[4]{4} = \sqrt[4]{8 \cdot 4}$ $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

$= \sqrt[4]{32}$ Find the greatest perfect fourth power that is a factor of 32.

$= \sqrt[4]{16 \cdot 2}$ $2^4 = 16$, so 16 is a perfect fourth power and is the greatest perfect fourth power that is a factor of 32.

$= \sqrt[4]{16} \cdot \sqrt[4]{2}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 2\sqrt[4]{2}$ $\sqrt[4]{16} = 2$

c. $\sqrt[4]{\dfrac{81}{16}} = \dfrac{\sqrt[4]{81}}{\sqrt[4]{16}}$ $\quad\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$= \dfrac{3}{2}$ $\quad\sqrt[4]{81} = 3$ because $3^4 = 81$ and $\sqrt[4]{16} = 2$ because $2^4 = 16$.

Check Point 8 Simplify:

a. $\sqrt[3]{40}$ **b.** $\sqrt[5]{8} \cdot \sqrt[5]{8}$ **c.** $\sqrt[3]{\dfrac{125}{27}}$.

We have seen that adding and subtracting square roots often involves simplifying terms. The same idea applies to adding and subtracting higher roots.

EXAMPLE 9 Combining Cube Roots

Subtract: $5\sqrt[3]{16} - 11\sqrt[3]{2}$.

SOLUTION

$5\sqrt[3]{16} - 11\sqrt[3]{2}$

$= 5\sqrt[3]{8 \cdot 2} - 11\sqrt[3]{2}$ Factor 16. 8 is the greatest perfect cube factor: $2^3 = 8$ and $\sqrt[3]{8} = 2$.

$= 5 \cdot 2\sqrt[3]{2} - 11\sqrt[3]{2}$ $\quad\sqrt[3]{8 \cdot 2} = \sqrt[3]{8}\sqrt[3]{2} = 2\sqrt[3]{2}$

$= 10\sqrt[3]{2} - 11\sqrt[3]{2}$ Multiply: $5 \cdot 2 = 10$.

$= (10 - 11)\sqrt[3]{2}$ Apply the distributive property.

$= -1\sqrt[3]{2}$ or $-\sqrt[3]{2}$ Simplify.

Check Point 9 Subtract: $3\sqrt[3]{81} - 4\sqrt[3]{3}$.

8 Understand and use rational exponents.

Rational Exponents

We define rational exponents so that their properties are the same as the properties for integer exponents. For example, we know that exponents are multiplied when an exponential expression is raised to a power. For this to be true,

$$\left(7^{\frac{1}{2}}\right)^2 = 7^{\frac{1}{2}\cdot 2} = 7^1 = 7.$$

We also know that

$$(\sqrt{7})^2 = \sqrt{7} \cdot \sqrt{7} = \sqrt{49} = 7.$$

Can you see that the square of both $7^{\frac{1}{2}}$ and $\sqrt{7}$ is 7? It is reasonable to conclude that

$$7^{\frac{1}{2}} \quad \text{means} \quad \sqrt{7}.$$

We can generalize the fact that $7^{\frac{1}{2}}$ means $\sqrt{7}$ with the following definition:

The Definition of $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ represents a real number, where $n \geq 2$ is an integer, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

The denominator of the rational exponent is the radical's index.

Furthermore,

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{a}}, \quad a \neq 0.$$

TECHNOLOGY

This graphing utility screen shows that

$$\sqrt{64} = 8 \text{ and } 64^{\frac{1}{2}} = 8.$$

EXAMPLE 10 Using the Definition of $a^{\frac{1}{n}}$

Simplify:

a. $64^{\frac{1}{2}}$ **b.** $125^{\frac{1}{3}}$ **c.** $-16^{\frac{1}{4}}$ **d.** $(-27)^{\frac{1}{3}}$ **e.** $64^{-\frac{1}{3}}$.

SOLUTION

a. $64^{\frac{1}{2}} = \sqrt{64} = 8$

b. $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

The denominator is the index.

c. $-16^{\frac{1}{4}} = -(\sqrt[4]{16}) = -2$

The base is 16 and the negative sign is not affected by the exponent.

d. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

Parentheses show that the base is −27 and that the negative sign is affected by the exponent.

e. $64^{-\frac{1}{3}} = \dfrac{1}{64^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{64}} = \dfrac{1}{4}$

• • •

Check Point 10 Simplify:

a. $25^{\frac{1}{2}}$ **b.** $8^{\frac{1}{3}}$ **c.** $-81^{\frac{1}{4}}$ **d.** $(-8)^{\frac{1}{3}}$ **e.** $27^{-\frac{1}{3}}$.

In Example 10 and Check Point 10, each rational exponent had a numerator of 1. If the numerator is some other integer, we still want to multiply exponents when raising a power to a power. For this reason,

$$a^{\frac{2}{3}} = \left(a^{\frac{1}{3}}\right)^2 \quad \text{and} \quad a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}}.$$

This means $(\sqrt[3]{a})^2$. This means $\sqrt[3]{a^2}$.

Thus,

$$a^{\frac{2}{3}} = (\sqrt[3]{a})^2 = \sqrt[3]{a^2}.$$

Do you see that the denominator, 3, of the rational exponent is the same as the index of the radical? The numerator, 2, of the rational exponent serves as an exponent in each of the two radical forms. We generalize these ideas with the following definition:

The Definition of $a^{\frac{m}{n}}$

If $\sqrt[n]{a}$ represents a real number and $\frac{m}{n}$ is a positive rational number, $n \geq 2$, then

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

Also,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Furthermore, if $a^{-\frac{m}{n}}$ is a nonzero real number, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}.$$

The first form of the definition of $a^{\frac{m}{n}}$, shown again below, involves taking the root first. This form is often preferable because smaller numbers are involved. Notice that the rational exponent consists of two parts, indicated by the following voice balloons:

The numerator is the exponent.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

The denominator is the radical's index.

EXAMPLE 11 Using the Definition of $a^{\frac{m}{n}}$

Simplify:

a. $27^{\frac{2}{3}}$ **b.** $9^{\frac{3}{2}}$ **c.** $81^{-\frac{3}{4}}$.

SOLUTION

a. $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$

b. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$

c. $81^{-\frac{3}{4}} = \dfrac{1}{81^{\frac{3}{4}}} = \dfrac{1}{(\sqrt[4]{81})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

TECHNOLOGY

Here are the calculator keystroke sequences for $81^{-\frac{3}{4}}$:

Many Scientific Calculators

81 $\boxed{y^x}$ $\boxed{(}$ 3 $\boxed{+/-}$ $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{=}$

Many Graphing Calculators

81 $\boxed{\wedge}$ $\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{ENTER}}$.

The parentheses around the exponent may not be necessary on your graphing calculator. However, their use here illustrates the correct order of operations for this computation. Try the keystrokes with the parentheses and then again without them. If you get different results, then you should always enclose rational exponents in parentheses.

Check Point 11 Simplify:

a. $27^{\frac{4}{3}}$ **b.** $4^{\frac{3}{2}}$ **c.** $32^{-\frac{2}{5}}$.

Properties of exponents can be applied to expressions containing rational exponents.

EXAMPLE 12 Simplifying Expressions with Rational Exponents

Simplify using properties of exponents:

a. $\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right)$ **b.** $\dfrac{32x^{\frac{5}{3}}}{16x^{\frac{3}{4}}}$.

SOLUTION

a. $\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right) = 5 \cdot 7x^{\frac{1}{2}} \cdot x^{\frac{3}{4}}$ Group numerical factors and group variable factors with the same base.

$= 35x^{\frac{1}{2}+\frac{3}{4}}$ When multiplying expressions with the same base, add the exponents.

$= 35x^{\frac{5}{4}}$ $\frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$

b. $\dfrac{32x^{\frac{5}{3}}}{16x^{\frac{3}{4}}} = \left(\dfrac{32}{16}\right)\left(\dfrac{x^{\frac{5}{3}}}{x^{\frac{3}{4}}}\right)$ Group numerical factors and group variable factors with the same base.

$= 2x^{\frac{5}{3}-\frac{3}{4}}$ When dividing expressions with the same base, subtract the exponents.

$= 2x^{\frac{11}{12}}$ $\frac{5}{3} - \frac{3}{4} = \frac{20}{12} - \frac{9}{12} = \frac{11}{12}$

Check Point 12 Simplify using properties of exponents:

a. $\left(2x^{\frac{4}{3}}\right)\left(5x^{\frac{8}{3}}\right)$ **b.** $\dfrac{20x^4}{5x^{\frac{3}{2}}}$.

Rational exponents are sometimes useful for simplifying radicals by reducing the index.

EXAMPLE 13 Reducing the Index of a Radical

Simplify: $\sqrt[9]{x^3}$.

SOLUTION

$$\sqrt[9]{x^3} = x^{\frac{3}{9}} = x^{\frac{1}{3}} = \sqrt[3]{x}$$

Check Point 13 Simplify: $\sqrt[6]{x^3}$.

Blitzer Bonus || A Radical Idea: Time Is Relative

The Persistence of Memory (1931), Salvador Dali:. © 2011 MOMA/ARS.

What does travel in space have to do with radicals? Imagine that in the future we will be able to travel at velocities approaching the speed of light (approximately 186,000 miles per second). According to Einstein's theory of special relativity, time would pass more quickly on Earth than it would in the moving spaceship. The special-relativity equation

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

gives the aging rate of an astronaut, R_a, relative to the aging rate of a friend, R_f, on Earth. In this formula, v is the astronaut's speed and c is the speed of light. As the astronaut's speed approaches the speed of light, we can substitute c for v.

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2}$$ Einstein's equation gives the aging rate of an astronaut, R_a, relative to the aging rate of a friend, R_f, on Earth.

$$R_a = R_f\sqrt{1 - \left(\frac{c}{c}\right)^2}$$ The velocity, v, is approaching the speed of light, c, so let $v = c$.

$$= R_f\sqrt{1 - 1}$$ $\left(\frac{c}{c}\right)^2 = 1^2 = 1 \cdot 1 = 1$

$$= R_f\sqrt{0}$$ Simplify the radicand: $1 - 1 = 0$.

$$= R_f \cdot 0$$ $\sqrt{0} = 0$

$$= 0$$ Multiply: $R_f \cdot 0 = 0$.

Close to the speed of light, the astronaut's aging rate, R_a, relative to a friend, R_f, on Earth is nearly 0. What does this mean? As we age here on Earth, the space traveler would barely get older. The space traveler would return to an unknown futuristic world in which friends and loved ones would be long gone.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. The symbol $\sqrt{}$ is used to denote the nonnegative, or ________, square root of a number.
2. $\sqrt{64} = 8$ because _____ $= 64$.
3. $\sqrt{a^2} =$ ______
4. The product rule for square roots states that if a and b are nonnegative, then $\sqrt{ab} =$ _______.
5. The quotient rule for square roots states that if a and b are nonnegative and $b \neq 0$, then $\sqrt{\frac{a}{b}} =$ _______.
6. $8\sqrt{3} + 10\sqrt{3} =$ ______
7. $\sqrt{3} + \sqrt{75} = \sqrt{3} + \sqrt{25 \cdot 3} = \sqrt{3} +$ ____ $\sqrt{3} =$ ______
8. The conjugate of $7 + \sqrt{3}$ is _______.
9. We rationalize the denominator of $\frac{5}{\sqrt{10} - \sqrt{2}}$ by multiplying the numerator and denominator by ________.
10. In the expression $\sqrt[3]{64}$, the number 3 is called the _______ and the number 64 is called the ________.
11. $\sqrt[5]{-32} = -2$ because _______ $= -32$.
12. If n is odd, $\sqrt[n]{a^n} =$ _____. If n is even, $\sqrt[n]{a^n} =$ _____.
13. $a^{\frac{1}{n}} =$ ____
14. $16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = (\_\_)^3 = \_\_$

EXERCISE SET P.3

Practice Exercises

Evaluate each expression in Exercises 1–12, or indicate that the root is not a real number.

1. $\sqrt{36}$
2. $\sqrt{25}$
3. $-\sqrt{36}$
4. $-\sqrt{25}$
5. $\sqrt{-36}$
6. $\sqrt{-25}$
7. $\sqrt{25 - 16}$
8. $\sqrt{144 + 25}$
9. $\sqrt{25} - \sqrt{16}$
10. $\sqrt{144} + \sqrt{25}$
11. $\sqrt{(-13)^2}$
12. $\sqrt{(-17)^2}$

Use the product rule to simplify the expressions in Exercises 13–22. In Exercises 17–22, assume that variables represent nonnegative real numbers.

13. $\sqrt{50}$
14. $\sqrt{27}$
15. $\sqrt{45x^2}$
16. $\sqrt{125x^2}$
17. $\sqrt{2x} \cdot \sqrt{6x}$
18. $\sqrt{10x} \cdot \sqrt{8x}$
19. $\sqrt{x^3}$
20. $\sqrt{y^3}$
21. $\sqrt{2x^2} \cdot \sqrt{6x}$
22. $\sqrt{6x} \cdot \sqrt{3x^2}$

Use the quotient rule to simplify the expressions in Exercises 23–32. Assume that $x > 0$.

23. $\sqrt{\frac{1}{81}}$
24. $\sqrt{\frac{1}{49}}$
25. $\sqrt{\frac{49}{16}}$
26. $\sqrt{\frac{121}{9}}$
27. $\frac{\sqrt{48x^3}}{\sqrt{3x}}$
28. $\frac{\sqrt{72x^3}}{\sqrt{8x}}$
29. $\frac{\sqrt{150x^4}}{\sqrt{3x}}$
30. $\frac{\sqrt{24x^4}}{\sqrt{3x}}$
31. $\frac{\sqrt{200x^3}}{\sqrt{10x^{-1}}}$
32. $\frac{\sqrt{500x^3}}{\sqrt{10x^{-1}}}$

In Exercises 33–44, add or subtract terms whenever possible.

33. $7\sqrt{3} + 6\sqrt{3}$
34. $8\sqrt{5} + 11\sqrt{5}$
35. $6\sqrt{17x} - 8\sqrt{17x}$
36. $4\sqrt{13x} - 6\sqrt{13x}$
37. $\sqrt{8} + 3\sqrt{2}$
38. $\sqrt{20} + 6\sqrt{5}$
39. $\sqrt{50x} - \sqrt{8x}$
40. $\sqrt{63x} - \sqrt{28x}$
41. $3\sqrt{18} + 5\sqrt{50}$
42. $4\sqrt{12} - 2\sqrt{75}$
43. $3\sqrt{8} - \sqrt{32} + 3\sqrt{72} - \sqrt{75}$
44. $3\sqrt{54} - 2\sqrt{24} - \sqrt{96} + 4\sqrt{63}$

In Exercises 45–54, rationalize the denominator.

45. $\frac{1}{\sqrt{7}}$
46. $\frac{2}{\sqrt{10}}$
47. $\frac{\sqrt{2}}{\sqrt{5}}$
48. $\frac{\sqrt{7}}{\sqrt{3}}$
49. $\frac{13}{3 + \sqrt{11}}$
50. $\frac{3}{3 + \sqrt{7}}$
51. $\frac{7}{\sqrt{5} - 2}$
52. $\frac{5}{\sqrt{3} - 1}$
53. $\frac{6}{\sqrt{5} + \sqrt{3}}$
54. $\frac{11}{\sqrt{7} - \sqrt{3}}$

Evaluate each expression in Exercises 55–66, or indicate that the root is not a real number.

55. $\sqrt[3]{125}$
56. $\sqrt[3]{8}$
57. $\sqrt[3]{-8}$
58. $\sqrt[3]{-125}$
59. $\sqrt[4]{-16}$
60. $\sqrt[4]{-81}$
61. $\sqrt[4]{(-3)^4}$
62. $\sqrt[4]{(-2)^4}$
63. $\sqrt[5]{(-3)^5}$
64. $\sqrt[5]{(-2)^5}$
65. $\sqrt[5]{-\frac{1}{32}}$
66. $\sqrt[6]{\frac{1}{64}}$

Simplify the radical expressions in Exercises 67–74 if possible.

67. $\sqrt[3]{32}$
68. $\sqrt[3]{150}$
69. $\sqrt[3]{x^4}$
70. $\sqrt[3]{x^5}$
71. $\sqrt[3]{9} \cdot \sqrt[3]{6}$
72. $\sqrt[3]{12} \cdot \sqrt[3]{4}$
73. $\frac{\sqrt[5]{64x^6}}{\sqrt[5]{2x}}$
74. $\frac{\sqrt[4]{162x^5}}{\sqrt[4]{2x}}$

In Exercises 75–82, add or subtract terms whenever possible.

75. $4\sqrt[5]{2} + 3\sqrt[5]{2}$ **76.** $6\sqrt[5]{3} + 2\sqrt[5]{3}$

77. $5\sqrt[3]{16} + \sqrt[3]{54}$ **78.** $3\sqrt[3]{24} + \sqrt[3]{81}$

79. $\sqrt[3]{54xy^3} - y\sqrt[3]{128x}$

80. $\sqrt[3]{24xy^3} - y\sqrt[3]{81x}$

81. $\sqrt{2} + \sqrt[3]{8}$ **82.** $\sqrt{3} + \sqrt[3]{15}$

In Exercises 83–90, evaluate each expression without using a calculator.

83. $36^{\frac{1}{2}}$ **84.** $121^{\frac{1}{2}}$

85. $8^{\frac{1}{3}}$ **86.** $27^{\frac{1}{3}}$

87. $125^{\frac{2}{3}}$ **88.** $8^{\frac{2}{3}}$

89. $32^{-\frac{4}{5}}$ **90.** $16^{-\frac{5}{2}}$

In Exercises 91–100, simplify using properties of exponents.

91. $\left(7x^{\frac{1}{3}}\right)\left(2x^{\frac{1}{4}}\right)$ **92.** $\left(3x^{\frac{2}{3}}\right)\left(4x^{\frac{3}{4}}\right)$

93. $\dfrac{20x^{\frac{1}{2}}}{5x^{\frac{1}{4}}}$ **94.** $\dfrac{72x^{\frac{3}{4}}}{9x^{\frac{1}{3}}}$

95. $\left(x^{\frac{2}{3}}\right)^3$ **96.** $\left(x^{\frac{4}{5}}\right)^5$

97. $(25x^4y^6)^{\frac{1}{2}}$ **98.** $(125x^9y^6)^{\frac{1}{3}}$

99. $\dfrac{\left(3y^{\frac{1}{4}}\right)^3}{y^{\frac{1}{12}}}$ **100.** $\dfrac{\left(2y^{\frac{1}{5}}\right)^4}{y^{\frac{3}{10}}}$

In Exercises 101–108, simplify by reducing the index of the radical.

101. $\sqrt[4]{5^2}$ **102.** $\sqrt[4]{7^2}$

103. $\sqrt[3]{x^6}$ **104.** $\sqrt[4]{x^{12}}$

105. $\sqrt[6]{x^4}$ **106.** $\sqrt[9]{x^6}$

107. $\sqrt[9]{x^6y^3}$ **108.** $\sqrt[12]{x^4y^8}$

Practice Plus

In Exercises 109–110, evaluate each expression.

109. $\sqrt[3]{\sqrt[4]{16} + \sqrt{625}}$

110. $\sqrt[3]{\sqrt{\sqrt{169} + \sqrt{9}} + \sqrt{\sqrt[3]{1000} + \sqrt[3]{216}}}$

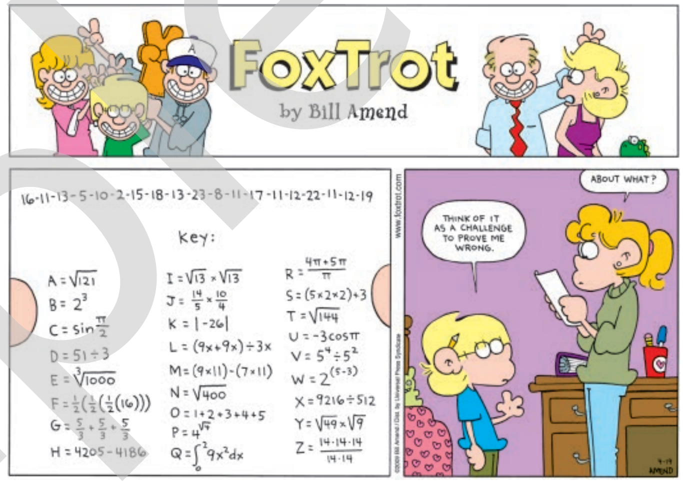
In Exercises 111–114, simplify each expression. Assume that all variables represent positive numbers.

111. $(49x^{-2}y^4)^{-\frac{1}{2}}\left(xy^{\frac{1}{2}}\right)$ **112.** $(8x^{-6}y^3)^{\frac{1}{3}}\left(x^{\frac{5}{6}}y^{-\frac{1}{3}}\right)^6$

113. $\left(\dfrac{x^{-\frac{5}{4}}y^{\frac{1}{3}}}{x^{-\frac{3}{4}}}\right)^{-6}$ **114.** $\left(\dfrac{x^{\frac{1}{2}}y^{-\frac{7}{4}}}{y^{-\frac{5}{4}}}\right)^{-4}$

Application Exercises

115. The popular comic strip *FoxTrot* follows the off-the-wall lives of the Fox family. Youngest son Jason is forever obsessed by his love of math. In the math-themed strip shown at the top of the next column, Jason shares his opinion in a coded message about the mathematical abilities of his sister Paige.



Solve problems A through Z in the left panel. Then decode Jason Fox's message involving his opinion about the mathematical abilities of his sister Paige shown on the first line.

Hints: Here is the solution for problem C and partial solutions for problems Q and U.

These are from trigonometry.

This is from calculus.

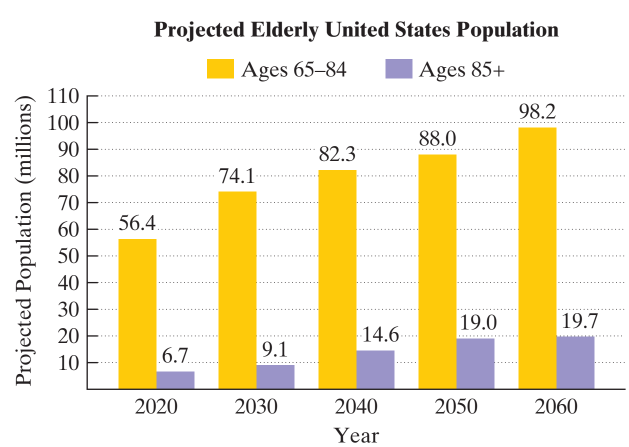
$$C = \sin\frac{\pi}{2} = \sin 90° = 1$$

$$Q = \int_0^2 9x^2dx = 3x^3\Big|_0^2 = 3\cdot2^3 - 3\cdot0^3 = \_\_\_\_$$

$$U = -3\cos\pi = -3\cos 180° = -3(-1) = \_\_\_\_$$

Note: The comic strip *FoxTrot* is now printed in more than one thousand newspapers. What made cartoonist Bill Amend, a college physics major, put math in the comic? "I always try to use math in the strip to make the joke accessible to anyone," he said. "But if you understand math, hopefully you'll like it that much more!" We highly recommend the math humor in Amend's *FoxTrot* collection *Math, Science, and Unix Underpants* (Andrews McMeel Publishing, 2009).

116. America is getting older. The graph shows the projected elderly U.S. population for ages 65–84 and for ages 85 and older.



Source: U.S. Census Bureau

The formula $E = 5.8\sqrt{x} + 56.4$ models the projected number of elderly Americans ages 65–84, E, in millions, x years after 2020.

a. Use the formula to find the projected increase in the number of Americans ages 65–84, in millions, from 2030 to 2060. Express this difference in simplified radical form.

b. Use a calculator and write your answer in part (a) to the nearest tenth. Does this rounded decimal overestimate or underestimate the difference in the projected data shown by the bar graph on the previous page? By how much?

117. The early Greeks believed that the most pleasing of all rectangles were **golden rectangles**, whose ratio of width to height is

$$\frac{w}{h} = \frac{2}{\sqrt{5} - 1}.$$

The Parthenon at Athens fits into a golden rectangle once the triangular pediment is reconstructed.

Rationalize the denominator of the golden ratio. Then use a calculator and find the ratio of width to height, correct to the nearest hundredth, in golden rectangles.

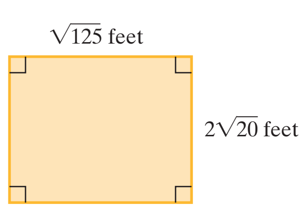
118. Use Einstein's special-relativity equation

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2},$$

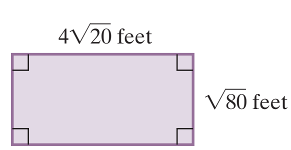
described in the Blitzer Bonus on page 47, to solve this exercise. You are moving at 90% of the speed of light. Substitute $0.9c$ for v, your velocity, in the equation. What is your aging rate, correct to two decimal places, relative to a friend on Earth? If you are gone for 44 weeks, approximately how many weeks have passed for your friend?

The perimeter, P, of a rectangle with length l and width w is given by the formula $P = 2l + 2w$. The area, A, is given by the formula $A = lw$. In Exercises 119–120, use these formulas to find the perimeter and area of each rectangle. Express answers in simplified radical form. Remember that perimeter is measured in linear units, such as feet or meters, and area is measured in square units, such as square feet, ft^2, or square meters, m^2.

119.



120.



Explaining the Concepts

121. Explain how to simplify $\sqrt{10} \cdot \sqrt{5}$.

122. Explain how to add $\sqrt{3} + \sqrt{12}$.

123. Describe what it means to rationalize a denominator. Use both $\frac{1}{\sqrt{5}}$ and $\frac{1}{5 + \sqrt{5}}$ in your explanation.

124. What difference is there in simplifying $\sqrt[3]{(-5)^3}$ and $\sqrt[4]{(-5)^4}$?

125. What does $a^{\frac{m}{n}}$ mean?

126. Describe the kinds of numbers that have rational fifth roots.

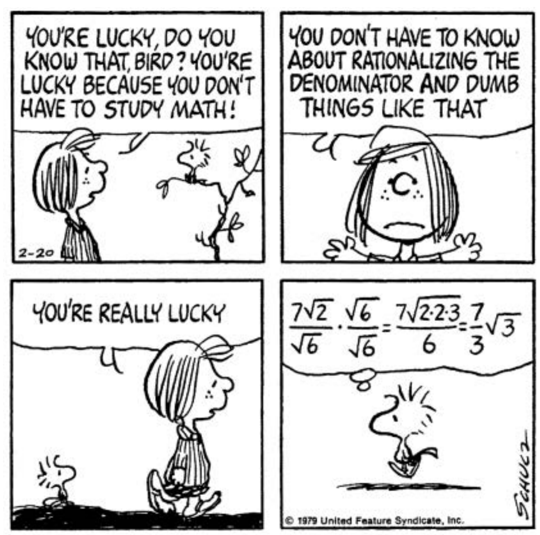
127. Why must a and b represent nonnegative numbers when we write $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$? Is it necessary to use this restriction in the case of $\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$? Explain.

128. Read the Blitzer Bonus on page 47. The future is now: You have the opportunity to explore the cosmos in a starship traveling near the speed of light. The experience will enable you to understand the mysteries of the universe in deeply personal ways, transporting you to unimagined levels of knowing and being. The downside: You return from your two-year journey to a futuristic world in which friends and loved ones are long gone. Do you explore space or stay here on Earth? What are the reasons for your choice?

Critical Thinking Exercises

Make Sense? *In Exercises 129–132, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

129. The joke in this Peanuts cartoon would be more effective if Woodstock had rationalized the denominator correctly in the last frame.



130. Using my calculator, I determined that $6^7 = 279{,}936$, so 6 must be a seventh root of 279,936.

131. I simplified the terms of $2\sqrt{20} + 4\sqrt{75}$, and then I was able to add the like radicals.

132. When I use the definition for $a^{\frac{m}{n}}$, I usually prefer to first raise a to the m power because smaller numbers are involved.

In Exercises 133–136, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

133. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 49$

134. $8^{-\frac{1}{3}} = -2$

135. The cube root of -8 is not a real number.

136. $\dfrac{\sqrt{20}}{8} = \dfrac{\sqrt{10}}{4}$

In Exercises 137–138, fill in each box to make the statement true.

137. $(5 + \sqrt{\square})(5 - \sqrt{\square}) = 22$

138. $\sqrt{\square x^{\square}} = 5x^7$

139. Find the exact value of $\sqrt{13 + \sqrt{2} + \dfrac{7}{3 + \sqrt{2}}}$ without the use of a calculator.

140. Place the correct symbol, $>$ or $<$, in the shaded area between the given numbers. *Do not use a calculator.* Then check your result with a calculator.

a. $3^{\frac{1}{2}} \; \square \; 3^{\frac{1}{3}}$ **b.** $\sqrt{7} + \sqrt{18} \; \square \; \sqrt{7 + 18}$

141. a. A mathematics professor recently purchased a birthday cake for her son with the inscription

$$\text{Happy}\left(2^{\frac{5}{2}} \cdot 2^{\frac{3}{4}} \div 2^{\frac{1}{4}}\right)\text{th Birthday.}$$

How old is the son?

b. The birthday boy, excited by the inscription on the cake, tried to wolf down the whole thing. Professor Mom, concerned about the possible metamorphosis of her son into a blimp, exclaimed, "Hold on! It is your birthday, so why not take $\dfrac{8^{-\frac{4}{3}} + 2^{-2}}{16^{-\frac{3}{4}} + 2^{-1}}$ of the cake? I'll eat half of what's left over." How much of the cake did the professor eat?

Preview Exercises

Exercises 142–144 will help you prepare for the material covered in the next section.

142. Multiply: $(2x^3y^2)(5x^4y^7)$.

143. Use the distributive property to multiply:

$$2x^4(8x^4 + 3x).$$

144. Simplify and express the answer in descending powers of x:

$$2x(x^2 + 4x + 5) + 3(x^2 + 4x + 5).$$

Section P.4 Polynomials

What am I supposed to learn?

After studying this section, you should be able to:

1. Understand the vocabulary of polynomials.
2. Add and subtract polynomials.
3. Multiply polynomials.
4. Use FOIL in polynomial multiplication.
5. Use special products in polynomial multiplication.
6. Perform operations with polynomials in several variables.

Old Dog . . . New Chicks

Can that be Axl, your author's yellow lab, sharing a special moment with a baby chick? And if it is (it is), what possible relevance can this have to polynomials? An answer is promised before you reach the Exercise Set. For now, we open the section by defining and describing polynomials.

How We Define Polynomials

More than 2 million people have tested their racial prejudice using an online version of the Implicit Association Test. Most groups' average scores fall between "slight" and "moderate" bias, but the differences among groups, by age and by political identification, are intriguing.

In this section's Exercise Set (Exercises 91 and 92), you will be working with models that measure bias:

$$S = 0.3x^3 - 2.8x^2 + 6.7x + 30$$
$$S = -0.03x^3 + 0.2x^2 + 2.3x + 24.$$

In each model, S represents the score on the Implicit Association Test. (Higher scores indicate stronger bias.) In the first model (see Exercise 91), x represents age group. In the second model (see Exercise 92), x represents political identification.

The algebraic expressions that appear on the right sides of the models are examples of *polynomials*. A **polynomial** is a single term or the sum of two or more terms containing variables with whole-number exponents. These particular polynomials each contain four terms. Equations containing polynomials are used in such diverse areas as science, business, medicine, psychology, and sociology. In this section, we review basic ideas about polynomials and their operations.

1 Understand the vocabulary of polynomials.

How We Describe Polynomials

Consider the polynomial

$$7x^3 - 9x^2 + 13x - 6.$$

We can express $7x^3 - 9x^2 + 13x - 6$ as

$$7x^3 + (-9x^2) + 13x + (-6).$$

The polynomial contains four terms. It is customary to write the terms in the order of descending powers of the variable. This is the **standard form** of a polynomial.

Some polynomials contain only one variable. Each term of such a polynomial in x is of the form ax^n. If $a \neq 0$, the **degree** of ax^n is n. For example, the degree of the term $7x^3$ is 3.

GREAT QUESTION!

Why doesn't the constant 0 have a degree?

We can express 0 in many ways, including $0x$, $0x^2$, and $0x^3$. It is impossible to assign a single exponent on the variable. This is why 0 has no defined degree.

The Degree of ax^n

If $a \neq 0$, the degree of ax^n is n. The degree of a nonzero constant is 0. The constant 0 has no defined degree.

Here is an example of a polynomial and the degree of each of its four terms:

$$6x^4 - 3x^3 - 2x - 5.$$

degree 4 | degree 3 | degree 1 | degree of nonzero constant: 0

Notice that the exponent on x for the term $2x$, meaning $2x^1$, is understood to be 1. For this reason, the degree of $2x$ is 1. You can think of -5 as $-5x^0$; thus, its degree is 0.

A polynomial is simplified when it contains no grouping symbols and no like terms. A simplified polynomial that has exactly one term is called a **monomial**. A **binomial** is a simplified polynomial that has two terms. A **trinomial** is a simplified polynomial with three terms. Simplified polynomials with four or more terms have no special names.

The **degree of a polynomial** is the greatest degree of all the terms of the polynomial. For example, $4x^2 + 3x$ is a binomial of degree 2 because the degree of the first term is 2, and the degree of the other term is less than 2. Also, $7x^5 - 2x^2 + 4$ is a trinomial of degree 5 because the degree of the first term is 5, and the degrees of the other terms are less than 5.

Up to now, we have used x to represent the variable in a polynomial. However, any letter can be used. For example,

- $7x^5 - 3x^3 + 8$ is a polynomial (in x) of degree 5. Because there are three terms, the polynomial is a trinomial.
- $6y^3 + 4y^2 - y + 3$ is a polynomial (in y) of degree 3. Because there are four terms, the polynomial has no special name.
- $z^7 + \sqrt{2}$ is a polynomial (in z) of degree 7. Because there are two terms, the polynomial is a binomial.

We can tie together the threads of our discussion with the formal definition of a polynomial in one variable. In this definition, the coefficients of the terms are represented by a_n (read "a sub n"), a_{n-1} (read "a sub n minus 1"), a_{n-2}, and so on. The small letters to the lower right of each a are called **subscripts** and are *not exponents*. Subscripts are used to distinguish one constant from another when a large and undetermined number of such constants are needed.

Definition of a Polynomial in x

A **polynomial in x** is an algebraic expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0,$$

where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$, and a_0 are real numbers, $a_n \neq 0$, and n is a nonnegative integer. The polynomial is of **degree n**, a_n is the **leading coefficient**, and a_0 is the **constant term**.

2 Add and subtract polynomials.

Adding and Subtracting Polynomials

Polynomials are added and subtracted by combining like terms. For example, we can combine the monomials $-9x^3$ and $13x^3$ using addition as follows:

$$-9x^3 + 13x^3 = (-9 + 13)x^3 = 4x^3.$$

These like terms both contain x to the third power.

Add coefficients and keep the same variable factor, x^3.

EXAMPLE 1 Adding and Subtracting Polynomials

Perform the indicated operations and simplify:

a. $(-9x^3 + 7x^2 - 5x + 3) + (13x^3 + 2x^2 - 8x - 6)$

b. $(7x^3 - 8x^2 + 9x - 6) - (2x^3 - 6x^2 - 3x + 9)$.

GREAT QUESTION!

Can I use a vertical format to add and subtract polynomials?

Yes. Arrange like terms in columns and combine vertically:

$$\begin{array}{rrrr} 7x^3 & -\ 8x^2 & +\ \ 9x & -\ \ 6 \\ -2x^3 & +\ 6x^2 & +\ \ 3x & -\ \ 9 \\ \hline 5x^3 & -\ 2x^2 & +\ 12x & -\ 15 \end{array}$$

The like terms can be combined by adding their coefficients and keeping the same variable factor.

SOLUTION

a. $(-9x^3 + 7x^2 - 5x + 3) + (13x^3 + 2x^2 - 8x - 6)$

$= (-9x^3 + 13x^3) + (7x^2 + 2x^2) + (-5x - 8x) + (3 - 6)$ Group like terms.

$= 4x^3 + 9x^2 + (-13x) + (-3)$ Combine like terms.

$= 4x^3 + 9x^2 - 13x - 3$ Simplify.

b. $(7x^3 - 8x^2 + 9x - 6) - (2x^3 - 6x^2 - 3x + 9)$ Rewrite subtraction as addition of the additive inverse.

Change the sign of each coefficient.

$= (7x^3 - 8x^2 + 9x - 6) + (-2x^3 + 6x^2 + 3x - 9)$

$= (7x^3 - 2x^3) + (-8x^2 + 6x^2) + (9x + 3x) + (-6 - 9)$ Group like terms.

$= 5x^3 + (-2x^2) + 12x + (-15)$ Combine like terms.

$= 5x^3 - 2x^2 + 12x - 15$ Simplify.

Check Point 1 Perform the indicated operations and simplify:

a. $(-17x^3 + 4x^2 - 11x - 5) + (16x^3 - 3x^2 + 3x - 15)$

b. $(13x^3 - 9x^2 - 7x + 1) - (-7x^3 + 2x^2 - 5x + 9)$.

3 Multiply polynomials.

Multiplying Polynomials

The product of two monomials is obtained by using properties of exponents. For example,

$$(-8x^6)(5x^3) = -8 \cdot 5x^{6+3} = -40x^9.$$

Multiply coefficients and add exponents.

GREAT QUESTION!

Because monomials with the same base and different exponents can be multiplied, can they also be added?

No. Don't confuse adding and multiplying monomials.

Addition:

$$5x^4 + 6x^4 = 11x^4$$

Multiplication:

$$\begin{aligned}(5x^4)(6x^4) &= (5 \cdot 6)(x^4 \cdot x^4) \\ &= 30x^{4+4} \\ &= 30x^8\end{aligned}$$

Only like terms can be added or subtracted, but unlike terms may be multiplied.

Addition:

$5x^4 + 3x^2$ cannot be simplified.

Multiplication:

$$\begin{aligned}(5x^4)(3x^2) &= (5 \cdot 3)(x^4 \cdot x^2) \\ &= 15x^{4+2} \\ &= 15x^6\end{aligned}$$

Furthermore, we can use the distributive property to multiply a monomial and a polynomial that is not a monomial. For example,

$$3x^4(2x^3 - 7x + 3) = 3x^4 \cdot 2x^3 - 3x^4 \cdot 7x + 3x^4 \cdot 3 = 6x^7 - 21x^5 + 9x^4.$$

Monomial

Trinomial

How do we multiply two polynomials if neither is a monomial? For example, consider

$$(2x + 3)(x^2 + 4x + 5).$$

Binomial

Trinomial

One way to perform $(2x + 3)(x^2 + 4x + 5)$ is to distribute $2x$ throughout the trinomial

$$2x(x^2 + 4x + 5)$$

and 3 throughout the trinomial

$$3(x^2 + 4x + 5).$$

Then combine the like terms that result.

Multiplying Polynomials When Neither Is a Monomial

Multiply each term of one polynomial by each term of the other polynomial. Then combine like terms.

EXAMPLE 2 Multiplying a Binomial and a Trinomial

Multiply: $(2x + 3)(x^2 + 4x + 5)$.

SOLUTION

$$(2x + 3)(x^2 + 4x + 5)$$

$$= 2x(x^2 + 4x + 5) + 3(x^2 + 4x + 5)$$ Multiply the trinomial by each term of the binomial.

$$= 2x \cdot x^2 + 2x \cdot 4x + 2x \cdot 5 + 3x^2 + 3 \cdot 4x + 3 \cdot 5$$ Use the distributive property.

$$= 2x^3 + 8x^2 + 10x + 3x^2 + 12x + 15$$ Multiply monomials: Multiply coefficients and add exponents.

$$= 2x^3 + 11x^2 + 22x + 15$$ Combine like terms: $8x^2 + 3x^2 = 11x^2$ and $10x + 12x = 22x$.

Another method for performing the multiplication is to use a vertical format similar to that used for multiplying whole numbers.

$$\begin{array}{rrrrl}
 & x^2 + & 4x + & 5 & \\
 & & 2x + & 3 & \\
\hline
 & 3x^2 + & 12x + & 15 & \quad 3(x^2 + 4x + 5) \\
2x^3 + & 8x^2 + & 10x & & \quad 2x(x^2 + 4x + 5) \\
\hline
2x^3 + & 11x^2 + & 22x + & 15 &
\end{array}$$

Write like terms in the same column.

Combine like terms.

Check Point 2 Multiply: $(5x - 2)(3x^2 - 5x + 4)$.

4 Use FOIL in polynomial multiplication.

The Product of Two Binomials: FOIL

Frequently, we need to find the product of two binomials. One way to perform this multiplication is to distribute each term in the first binomial through the second binomial. For example, we can find the product of the binomials $3x + 2$ and $4x + 5$ as follows:

$$\begin{aligned}(3x + 2)(4x + 5) &= 3x(4x + 5) + 2(4x + 5)\\ &= 3x(4x) + 3x(5) + 2(4x) + 2(5)\\ &= 12x^2 + 15x + 8x + 10.\end{aligned}$$

Distribute $3x$ over $4x + 5$.

Distribute 2 over $4x + 5$.

We'll combine these like terms later. For now, our interest is in how to obtain *each* of these four terms.

We can also find the product of $3x + 2$ and $4x + 5$ using a method called FOIL, which is based on our preceding work. Any two binomials can be quickly multiplied by using the FOIL method, in which **F** represents the product of the **first** terms in each binomial, **O** represents the product of the **outside** terms, **I** represents the product of the **inside** terms, and **L** represents the product of the **last**, or second, terms in each binomial. For example, we can use the FOIL method to find the product of the binomials $3x + 2$ and $4x + 5$ as follows:

first last

F O I L

$$(3x + 2)(4x + 5) = 12x^2 + 15x + 8x + 10$$

inside

outside

Product of First terms | Product of Outside terms | Product of Inside terms | Product of Last terms

$$= 12x^2 + 23x + 10$$ Combine like terms.

In general, here's how to use the FOIL method to find the product of $ax + b$ and $cx + d$:

Using the FOIL Method to Multiply Binomials

first last

F O I L

$$(ax + b)(cx + d) = ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d$$

inside

outside

Product of First terms | Product of Outside terms | Product of Inside terms | Product of Last terms

EXAMPLE 3 Using the FOIL Method

Multiply: $(3x + 4)(5x - 3)$.

SOLUTION

first last

F O I L

$$\begin{aligned}(3x + 4)(5x - 3) &= 3x \cdot 5x + 3x(-3) + 4 \cdot 5x + 4(-3)\\ &= 15x^2 - 9x + 20x - 12\\ &= 15x^2 + 11x - 12\end{aligned}$$ Combine like terms.

inside

outside

Check Point 3 Multiply: $(7x - 5)(4x - 3)$.

5 Use special products in polynomial multiplication.

Multiplying the Sum and Difference of Two Terms

We can use the FOIL method to multiply $A + B$ and $A - B$ as follows:

F O I L

$$(A + B)(A - B) = A^2 - AB + AB - B^2 = A^2 - B^2.$$

Notice that the outside and inside products have a sum of 0 and the terms cancel. The FOIL multiplication provides us with a quick rule for multiplying the sum and difference of two terms, referred to as a *special-product formula*.

The Product of the Sum and Difference of Two Terms

$$(A + B)(A - B) = A^2 - B^2$$

The product of the sum and the difference of the same two terms | is | the square of the first term minus the square of the second term.

EXAMPLE 4 Finding the Product of the Sum and Difference of Two Terms

Multiply:

a. $(4y + 3)(4y - 3)$ **b.** $(5a^4 + 6)(5a^4 - 6)$.

SOLUTION

Use the special-product formula shown.

$$(A + B)(A - B) = A^2 - B^2$$

First term squared − Second term squared = Product

a. $(4y + 3)(4y - 3) = (4y)^2 - 3^2 = 16y^2 - 9$

b. $(5a^4 + 6)(5a^4 - 6) = (5a^4)^2 - 6^2 = 25a^8 - 36$

Check Point 4 Multiply:

a. $(7x + 8)(7x - 8)$ **b.** $(2y^3 - 5)(2y^3 + 5)$.

The Square of a Binomial

Let us find $(A + B)^2$, the square of a binomial sum. To do so, we begin with the FOIL method and look for a general rule.

$$(A + B)^2 = (A + B)(A + B) = \underbrace{A \cdot A}_{\text{F}} + \underbrace{A \cdot B}_{\text{O}} + \underbrace{A \cdot B}_{\text{I}} + \underbrace{B \cdot B}_{\text{L}}$$
$$= A^2 + 2AB + B^2$$

This result implies the following rule, which is another example of a special-product formula:

The Square of a Binomial Sum

$$(A + B)^2 = A^2 + 2AB + B^2$$

The square of a binomial sum | is | first term squared | plus | 2 times the product of the terms | plus | last term squared.

GREAT QUESTION!

When finding $(x + 3)^2$, why can't I just write $x^2 + 3^2$, or $x^2 + 9$?

Caution! The square of a sum is *not* the sum of the squares.

$(A + B)^2 \neq A^2 + B^2$

The middle term $2AB$ is missing.

$(x + 3)^2 \neq x^2 + 9$

Incorrect!

Show that $(x + 3)^2$ and $x^2 + 9$ are not equal by substituting 5 for x in each expression and simplifying.

EXAMPLE 5 Finding the Square of a Binomial Sum

Multiply:

a. $(x + 3)^2$ **b.** $(3x + 7)^2$.

SOLUTION

Use the special-product formula shown.

$$(A + B)^2 = A^2 + 2AB + B^2$$

	(First Term)^2	+	2 · Product of the Terms	+	(Last Term)^2	= Product
a. $(x + 3)^2 =$	x^2	+	$2 \cdot x \cdot 3$	+	3^2	$= x^2 + 6x + 9$
b. $(3x + 7)^2 =$	$(3x)^2$	+	$2(3x)(7)$	+	7^2	$= 9x^2 + 42x + 49$

Check Point 5 Multiply:

a. $(x + 10)^2$ **b.** $(5x + 4)^2$.

A similar pattern occurs for $(A - B)^2$, the square of a binomial difference. Using the FOIL method on $(A - B)^2$, we obtain the following rule:

The Square of a Binomial Difference

$$(A - B)^2 = A^2 - 2AB + B^2$$

The square of a binomial difference | is | first term squared | minus | 2 times the product of the terms | plus | last term squared.

EXAMPLE 6 Finding the Square of a Binomial Difference

Multiply:

a. $(x - 4)^2$ **b.** $(5y - 6)^2$.

SOLUTION

Use the special-product formula shown.

$$(A - B)^2 = A^2 - 2AB + B^2$$

	(First Term)^2	−	2 · Product of the Terms	+	(Last Term)^2	= Product
a. $(x - 4)^2 =$	x^2	−	$2 \cdot x \cdot 4$	+	4^2	$= x^2 - 8x + 16$
b. $(5y - 6)^2 =$	$(5y)^2$	−	$2(5y)(6)$	+	6^2	$= 25y^2 - 60y + 36$

Check Point 6 Multiply:

a. $(x - 9)^2$ **b.** $(7x - 3)^2$.

Special Products

There are several products that occur so frequently that it's convenient to memorize the form, or pattern, of these formulas.

GREAT QUESTION!

Do I have to memorize the special products shown in the table on the right?

Not necessarily. Although it's convenient to memorize these forms, the FOIL method can be used on all five examples in the box. To cube $x + 4$, you can first square $x + 4$ using FOIL and then multiply this result by $x + 4$. We suggest memorizing these special forms because they let you multiply far more rapidly than using the FOIL method.

Special Products

Let A and B represent real numbers, variables, or algebraic expressions.

Special Product	**Example**
Sum and Difference of Two Terms	
$(A + B)(A - B) = A^2 - B^2$	$(2x + 3)(2x - 3) = (2x)^2 - 3^2$ $= 4x^2 - 9$
Squaring a Binomial	
$(A + B)^2 = A^2 + 2AB + B^2$	$(y + 5)^2 = y^2 + 2\cdot y\cdot 5 + 5^2$ $= y^2 + 10y + 25$
$(A - B)^2 = A^2 - 2AB + B^2$	$(3x - 4)^2$ $= (3x)^2 - 2\cdot 3x\cdot 4 + 4^2$ $= 9x^2 - 24x + 16$
Cubing a Binomial	
$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$	$(x + 4)^3$ $= x^3 + 3x^2(4) + 3x(4)^2 + 4^3$ $= x^3 + 12x^2 + 48x + 64$
$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$	$(x - 2)^3$ $= x^3 - 3x^2(2) + 3x(2)^2 - 2^3$ $= x^3 - 6x^2 + 12x - 8$

6 Perform operations with polynomials in several variables.

Polynomials in Several Variables

A **polynomial in two variables**, x and y, contains the sum of one or more monomials in the form ax^ny^m. The constant, a, is the **coefficient**. The exponents, n and m, represent whole numbers. The **degree** of the monomial ax^ny^m is $n + m$.

Here is an example of a polynomial in two variables:

The coefficients are 7, −17, 1, −6, and 9.

$$7x^2y^3 - 17x^4y^2 + xy - 6y^2 + 9.$$

Degree of monomial: $2 + 3 = 5$

Degree of monomial: $4 + 2 = 6$

Degree of monomial $(1x^1y^1)$: $1 + 1 = 2$

Degree of monomial $(-6x^0y^2)$: $0 + 2 = 2$

Degree of monomial $(9x^0y^0)$: $0 + 0 = 0$

The **degree of a polynomial in two variables** is the highest degree of all its terms. For the preceding polynomial, the degree is 6.

Polynomials containing two or more variables can be added, subtracted, and multiplied just like polynomials that contain only one variable. For example, we can add the monomials $-7xy^2$ and $13xy^2$ as follows:

$$-7xy^2 + 13xy^2 = (-7 + 13)xy^2 = 6xy^2.$$

These like terms both contain the variable factors x and y^2.

Add coefficients and keep the same variable factors, xy^2.

EXAMPLE 7 Subtracting Polynomials in Two Variables

Subtract:

$$(5x^3 - 9x^2y + 3xy^2 - 4) - (3x^3 - 6x^2y - 2xy^2 + 3).$$

SOLUTION

$$(5x^3 - 9x^2y + 3xy^2 - 4) - (3x^3 - 6x^2y - 2xy^2 + 3)$$

Change the sign of each coefficient.

$$= (5x^3 - 9x^2y + 3xy^2 - 4) + (-3x^3 + 6x^2y + 2xy^2 - 3)$$

Add the opposite of the polynomial being subtracted.

$$= (5x^3 - 3x^3) + (-9x^2y + 6x^2y) + (3xy^2 + 2xy^2) + (-4 - 3)$$

Group like terms.

$$= 2x^3 - 3x^2y + 5xy^2 - 7$$

Combine like terms by adding coefficients and keeping the same variable factors.

Check Point 7 Subtract: $(x^3 - 4x^2y + 5xy^2 - y^3) - (x^3 - 6x^2y + y^3)$.

EXAMPLE 8 Multiplying Polynomials in Two Variables

Multiply:

a. $(x + 4y)(3x - 5y)$ **b.** $(5x + 3y)^2$.

SOLUTION

We will perform the multiplication in part (a) using the FOIL method. We will multiply in part (b) using the formula for the square of a binomial sum, $(A + B)^2$.

a. $(x + 4y)(3x - 5y)$ Multiply these binomials using the FOIL method.

$$= \underbrace{(x)(3x)}_{F} + \underbrace{(x)(-5y)}_{O} + \underbrace{(4y)(3x)}_{I} + \underbrace{(4y)(-5y)}_{L}$$

$$= 3x^2 - 5xy + 12xy - 20y^2$$

$$= 3x^2 + 7xy - 20y^2$$ Combine like terms.

$$(A + B)^2 = A^2 + 2 \cdot A \cdot B + B^2$$

b. $(5x + 3y)^2 = (5x)^2 + 2(5x)(3y) + (3y)^2$

$$= 25x^2 + 30xy + 9y^2$$

Check Point 8 Multiply:

a. $(7x - 6y)(3x - y)$ **b.** $(2x + 4y)^2$.

Blitzer Bonus || Labrador Retrievers and Polynomial Multiplication

The color of a Labrador retriever is determined by its pair of genes. A single gene is inherited at random from each parent. The black-fur gene, B, is dominant. The yellow-fur gene, Y, is recessive. This means that labs with at least one black-fur gene (BB or BY) have black coats. Only labs with two yellow-fur genes (YY) have yellow coats.

Axl, your author's yellow lab, inherited his genetic makeup from two black BY parents.

Second BY parent, a black lab with a recessive yellow-fur gene

First BY parent, a black lab with a recessive yellow-fur gene

	B	Y
B	BB	BY
Y	BY	YY

The table shows the four possible combinations of color genes that BY parents can pass to their offspring.

Because YY is one of four possible outcomes, the probability that a yellow lab like Axl will be the offspring of these black parents is $\frac{1}{4}$.

The probabilities suggested by the table can be modeled by the expression $\left(\frac{1}{2}B + \frac{1}{2}Y\right)^2$.

$$\left(\frac{1}{2}B + \frac{1}{2}Y\right)^2 = \left(\frac{1}{2}B\right)^2 + 2\left(\frac{1}{2}B\right)\left(\frac{1}{2}Y\right) + \left(\frac{1}{2}Y\right)^2$$

$$= \frac{1}{4}BB \quad + \quad \frac{1}{2}BY \quad + \quad \frac{1}{4}YY$$

The probability of a black lab with two dominant black genes is $\frac{1}{4}$.

The probability of a black lab with a recessive yellow gene is $\frac{1}{2}$.

The probability of a yellow lab with two recessive yellow genes is $\frac{1}{4}$.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. A polynomial is a single term or the sum of two or more terms containing variables with exponents that are ___________ numbers.

2. It is customary to write the terms of a polynomial in the order of descending powers of the variable. This is called the ___________ form of a polynomial.

3. A simplified polynomial that has exactly one term is called a/an ___________.

4. A simplified polynomial that has two terms is called a/an ___________.

5. A simplified polynomial that has three terms is called a/an ___________.

6. If $a \neq 0$, the degree of ax^n is ____.

7. Polynomials are added by combining ______ terms.

8. To multiply $7x^3(4x^5 - 8x^2 + 6)$, use the ___________ property to multiply each term of the trinomial ___________ by the monomial _____.

9. To multiply $(5x + 3)(x^2 + 8x + 7)$, begin by multiplying each term of $x^2 + 8x + 7$ by ______. Then multiply each term of $x^2 + 8x + 7$ by ______. Then combine ______ terms.

10. When using the FOIL method to find $(x + 7)(3x + 5)$, the product of the first terms is ______, the product of the outside terms is ______, the product of the inside terms is ______, and the product of the last terms is ______.

11. $(A + B)(A - B) =$ ___________. The product of the sum and difference of the same two terms is the square of the first term ___________ the square of the second term.

12. $(A + B)^2 =$ ____________. The square of a binomial sum is the first term ________ plus 2 times the ____________ plus the last term ______.

13. $(A - B)^2 =$ ____________. The square of a binomial difference is the first term squared ______ 2 times the ____________ ______ the last term squared.

plus or minus?

14. If $a \neq 0$, the degree of ax^ny^m is ______.

EXERCISE SET P.4

Practice Exercises

In Exercises 1–4, is the algebraic expression a polynomial? If it is, write the polynomial in standard form.

1. $2x + 3x^2 - 5$
2. $2x + 3x^{-1} - 5$
3. $\frac{2x + 3}{x}$
4. $x^2 - x^3 + x^4 - 5$

In Exercises 5–8, find the degree of the polynomial.

5. $3x^2 - 5x + 4$
6. $-4x^3 + 7x^2 - 11$
7. $x^2 - 4x^3 + 9x - 12x^4 + 63$
8. $x^2 - 8x^3 + 15x^4 + 91$

In Exercises 9–14, perform the indicated operations. Write the resulting polynomial in standard form and indicate its degree.

9. $(-6x^3 + 5x^2 - 8x + 9) + (17x^3 + 2x^2 - 4x - 13)$
10. $(-7x^3 + 6x^2 - 11x + 13) + (19x^3 - 11x^2 + 7x - 17)$
11. $(17x^3 - 5x^2 + 4x - 3) - (5x^3 - 9x^2 - 8x + 11)$
12. $(18x^4 - 2x^3 - 7x + 8) - (9x^4 - 6x^3 - 5x + 7)$
13. $(5x^2 - 7x - 8) + (2x^2 - 3x + 7) - (x^2 - 4x - 3)$
14. $(8x^2 + 7x - 5) - (3x^2 - 4x) - (-6x^3 - 5x^2 + 3)$

In Exercises 15–58, find each product.

15. $(x + 1)(x^2 - x + 1)$
16. $(x + 5)(x^2 - 5x + 25)$
17. $(2x - 3)(x^2 - 3x + 5)$
18. $(2x - 1)(x^2 - 4x + 3)$
19. $(x + 7)(x + 3)$
20. $(x + 8)(x + 5)$
21. $(x - 5)(x + 3)$
22. $(x - 1)(x + 2)$
23. $(3x + 5)(2x + 1)$
24. $(7x + 4)(3x + 1)$
25. $(2x - 3)(5x + 3)$
26. $(2x - 5)(7x + 2)$
27. $(5x^2 - 4)(3x^2 - 7)$
28. $(7x^2 - 2)(3x^2 - 5)$
29. $(8x^3 + 3)(x^2 - 5)$
30. $(7x^3 + 5)(x^2 - 2)$
31. $(x + 3)(x - 3)$
32. $(x + 5)(x - 5)$
33. $(3x + 2)(3x - 2)$
34. $(2x + 5)(2x - 5)$
35. $(5 - 7x)(5 + 7x)$
36. $(4 - 3x)(4 + 3x)$
37. $(4x^2 + 5x)(4x^2 - 5x)$
38. $(3x^2 + 4x)(3x^2 - 4x)$
39. $(1 - y^5)(1 + y^5)$
40. $(2 - y^5)(2 + y^5)$
41. $(x + 2)^2$
42. $(x + 5)^2$
43. $(2x + 3)^2$
44. $(3x + 2)^2$
45. $(x - 3)^2$
46. $(x - 4)^2$
47. $(4x^2 - 1)^2$
48. $(5x^2 - 3)^2$
49. $(7 - 2x)^2$
50. $(9 - 5x)^2$
51. $(x + 1)^3$
52. $(x + 2)^3$
53. $(2x + 3)^3$
54. $(3x + 4)^3$
55. $(x - 3)^3$
56. $(x - 1)^3$
57. $(3x - 4)^3$
58. $(2x - 3)^3$

In Exercises 59–66, perform the indicated operations. Indicate the degree of the resulting polynomial.

59. $(5x^2y - 3xy) + (2x^2y - xy)$
60. $(-2x^2y + xy) + (4x^2y + 7xy)$
61. $(4x^2y + 8xy + 11) + (-2x^2y + 5xy + 2)$
62. $(7x^4y^2 - 5x^2y^2 + 3xy) + (-18x^4y^2 - 6x^2y^2 - xy)$
63. $(x^3 + 7xy - 5y^2) - (6x^3 - xy + 4y^2)$
64. $(x^4 - 7xy - 5y^3) - (6x^4 - 3xy + 4y^3)$
65. $(3x^4y^2 + 5x^3y - 3y) - (2x^4y^2 - 3x^3y - 4y + 6x)$
66. $(5x^4y^2 + 6x^3y - 7y) - (3x^4y^2 - 5x^3y - 6y + 8x)$

In Exercises 67–82, find each product.

67. $(x + 5y)(7x + 3y)$
68. $(x + 9y)(6x + 7y)$
69. $(x - 3y)(2x + 7y)$
70. $(3x - y)(2x + 5y)$
71. $(3xy - 1)(5xy + 2)$
72. $(7x^2y + 1)(2x^2y - 3)$
73. $(7x + 5y)^2$
74. $(9x + 7y)^2$
75. $(x^2y^2 - 3)^2$
76. $(x^2y^2 - 5)^2$
77. $(x - y)(x^2 + xy + y^2)$
78. $(x + y)(x^2 - xy + y^2)$
79. $(3x + 5y)(3x - 5y)$
80. $(7x + 3y)(7x - 3y)$
81. $(7xy^2 - 10y)(7xy^2 + 10y)$
82. $(3xy^2 - 4y)(3xy^2 + 4y)$

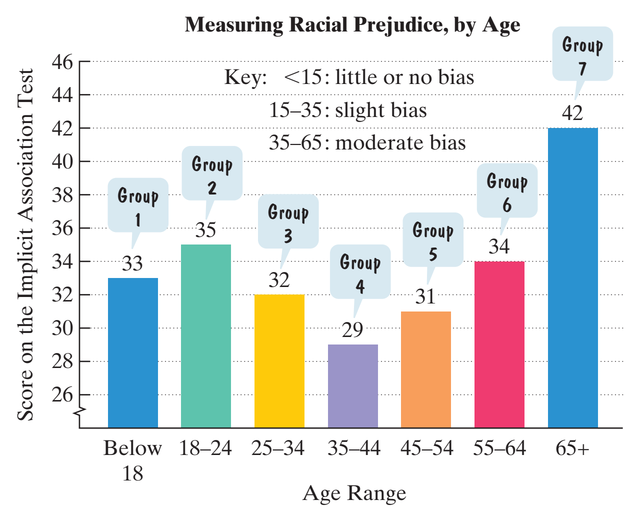
Practice Plus

In Exercises 83–90, perform the indicated operation or operations.

83. $(3x + 4y)^2 - (3x - 4y)^2$
84. $(5x + 2y)^2 - (5x - 2y)^2$
85. $(5x - 7)(3x - 2) - (4x - 5)(6x - 1)$
86. $(3x + 5)(2x - 9) - (7x - 2)(x - 1)$
87. $(2x + 5)(2x - 5)(4x^2 + 25)$
88. $(3x + 4)(3x - 4)(9x^2 + 16)$
89. $\frac{(2x - 7)^5}{(2x - 7)^3}$
90. $\frac{(5x - 3)^6}{(5x - 3)^4}$

Application Exercises

91. The bar graph shows the differences among age groups on the Implicit Association Test that measures levels of racial prejudice. Higher scores indicate stronger bias.



Source: The Race Implicit Association Test on the Project Implicit Demonstration Website